Coordinate geometry is the use of algebraic methods to study the geometry of straight lines and curves. In this chapter you will consider only straight lines. Curves will be studied later.

3.1 Cartesian coordinates

In your GCSE course you plotted points in two dimensions. This section revises that work and builds up the terminology required.

Learning objectives

After studying this chapter, you should be able to:

■ use the language of coordinate geometry
■ find the distance between two given points
■ find the coordinates of the mid-point of a line segment joining two given points
■ find, use and interpret the gradient of a line segment
■ know the relationship between the gradients for parallel and for perpendicular lines
■ find the equations of straight lines given (a) the gradient and y-intercept, (b) the gradient and a point, and (c) two points
■ verify given their coordinates, that points lie on a line
■ find the coordinates of a point of intersection of two lines
■ find the fourth vertex of a parallelogram given the other three.

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3.1 Cartesian coordinates

In your GCSE course you plotted points in two dimensions. This section revises that work and builds up the terminology required.

Named after René Descartes (1596–1650), a French mathematician.
In the diagram below, the point $P$ lies in the first quadrant.
Its distance from the $y$-axis is $a$.
Its distance from the $x$-axis is $b$.

We say that:
- the $x$-coordinate of $P$ is $a$.
- the $y$-coordinate of $P$ is $b$.

We write $P$ is the point $(a, b)$ or simply 'the point $P(a, b)$'.

The diagram below shows part of the Cartesian grid, where $L$ is the point $(-2, 3)$.

The $x$-coordinate of $L$ is $-2$ and the $y$-coordinate of $L$ is $3$.

Any point on the $x$-axis has its $x$-coordinate equal to $0$. $J$ and $K$ are points on the $x$-axis.
Worked example 3.1

1 Draw coordinate axes Ox and Oy and plot the points
   \(A(1, 2), B(-3, 3), C(1, -4)\) and \(D(-2, -4)\).

2 The line joining the points \(C\) and \(D\) crosses the \(y\)-axis at the
   point \(E\). Write down the coordinates of \(E\).

3 The line joining the points \(A\) and \(C\) crosses the \(x\)-axis at the
   point \(F\). Write down the coordinates of \(F\).

Solution

1 \(E\) is on the \(y\)-axis so its \(x\)-coordinate is 0. \(E(0, -4)\).

3 \(F\) is on the \(x\)-axis so its \(y\)-coordinate is 0. \(F(1, 0)\).

3.2 The distance between two points

The following example shows how you can use Pythagoras’ theorem which you studied as part of your GCSE course, in order to find the distance between two points.

Worked example 3.2

Find the distance \(AB\) where \(A\) is the point \((1, 2)\) and \(B\) is the
point \((4, 4)\).

Solution

First plot the points, join them with a line and make a
right-angled triangle \(ABC\).

The distance \(AC = 4 - 1 = 3\).

The distance \(BC = 4 - 2 = 2\).

Using Pythagoras’ theorem \(AB^2 = 3^2 + 2^2\).

\[AB = \sqrt{13}\]
We can generalise the method of the previous example to find a formula for the distance between the points \(P(x_1, y_1)\) and \(Q(x_2, y_2)\).

The distance \(PR = x_2 - x_1\).
The distance \(QR = y_2 - y_1\).

Using Pythagoras’ theorem
\[
PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

**Worked example 3.3**

The point \(R\) has coordinates \((-1, 2)\) and the point \(S\) has coordinates \((5, -6)\).

(a) Find the distance \(RS\).

(b) The point \(T\) has coordinates \((0, 9)\). Show that \(RT\) has length \(k\sqrt{2}\), where \(k\) is an integer.

**Solution**

(a) For points \(R\) and \(S\):
The difference between the \(x\)-coordinates is \(5 - (-1) = 6\)
The difference between the \(y\)-coordinates is \(-6 - 2 = -8\).

\[
RS = \sqrt{6^2 + (-8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10
\]

(b) \(RT = \sqrt{(1)^2 + (7)^2} = \sqrt{1 + 49} = \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}\)

**Worked example 3.4**

The distance \(MN\) is 5, where \(M\) is the point \((4, -2)\) and \(N\) is the point \((a, 2a)\). Find the two possible values of the constant \(a\).
Solution
The difference between the x-coordinates is \( a = 4 \).
The difference between the y-coordinates is \( 2a^2 = 2 \).

\[
MN^2 = (a - 4)^2 + (2a + 2)^2 = a^2 - 8a + 16 + 4a^2 + 8a + 4 = 5a^2 + 20
\]
\[
\Rightarrow a^2 = 1
\]
\[
\Rightarrow a = \pm 1.
\]

EXERCISE 3A
1 Find the lengths of the line segments joining:
   (a) \((0, 0)\) and \((3, 4)\),
   (b) \((1, 2)\) and \((5, 3)\),
   (c) \((0, 4)\) and \((5, 1)\),
   (d) \((-3, 1)\) and \((-1, 6)\),
   (e) \((4, -2)\) and \((3, 0)\),
   (f) \((-3, 2)\) and \((6, 1)\),
   (g) \((-2, 7)\) and \((3, -1)\),
   (h) \((-2, 0)\) and \((6, -3)\),
   (i) \((-1.5, 0)\) and \((3.5, 0)\),
   (j) \((2.5, 4)\) and \((1, 6)\),
   (k) \((8, 0)\) and \((2, 2.5)\),
   (l) \((-3.5, 2)\) and \((4, -8)\).

2 Calculate the lengths of the sides of the triangle \( ABC \) and hence determine whether or not the triangle is right-angled:
   (a) \(A(0, 0)\), \(B(0, 6)\), \(C(4, 3)\),
   (b) \(A(3, 0)\), \(B(1, 8)\), \(C(-7, 6)\),
   (c) \(A(1, 2)\), \(B(3, 4)\), \(C(0, 7)\).

3 The vertices of a triangle are \( A(1, 5)\), \( B(0, -2)\) and \( C(4, 2)\). By writing each of the lengths of the sides as a multiple of \( \sqrt{2} \), show that the sum of the lengths of two of the sides is three times the length of the third side.

4 The distance between the two points \( A(6, 2p)\) and \( B(p, -3)\) is \(5\sqrt{5}\). Find the possible values of \( p \).

5 The vertices of a triangle are \( P(1, 3)\), \( Q(-2, 0)\) and \( R(4, 0)\).
   (a) Find the lengths of the sides of triangle \( PQR \).
   (b) Show that angle \( QPR = 90^\circ \).
   (c) The line of symmetry of triangle \( PQR \) meets the x-axis at point \( S \). Write down the coordinates of \( S \).
   (d) The point \( T \) is such that \( PQTR \) is a square. Find the coordinates of \( T \).
3.3 The coordinates of the mid-point of a line segment joining two known points

From the diagram you can see that the mid-point of the line segment joining (0, 0) to (6, 0) is (3, 0) or 
\[ \left( \frac{0 + 6}{2}, 0 \right) \] 
the mid-point of the line segment joining (0, 0) to (0, −4) is (0, −2) or 
\[ \left( 0, \frac{0 + (-4)}{2} \right) \] 
and the mid-point of the line segment joining (0, −4) to (6, 0) is (3, −2) or 
\[ \left( \frac{0 + 6 - 4}{2}, 0 \right) \] 

Going from \( P(1, 2) \) to \( Q(5, 8) \) you move 4 units horizontally and then 6 vertically.

If \( M \) is the mid-point of \( PQ \) then the journey is halved so to go from \( P(1, 2) \) to \( M \) you move 2 (half of 4) horizontally and then 3 (half of 6) vertically.

So \( M \) is the point \((1 + 2, 2 + 3)\) or \(M(3, 5)\) or \(M\left( \frac{1 + 5}{2}, \frac{2 + 8}{2} \right)\).

Note that \( x = \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2) \) which is the x-coordinate of \( M \).

In general, the coordinates of the mid-point of the line segment joining \((x_1, y_1)\) and \((x_2, y_2)\) are 
\[ \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \]
Worked example 3.5

M is the mid-point of the line segment joining A(1, −2) and B(3, 5).

(a) Find the coordinates of M.

(b) M is also the mid-point of the line segment CD where C(1, 4). Find the coordinates of D.

Solution

(a) Using \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \), M is the point \( \left( \frac{1 + 3}{2}, \frac{-2 + 5}{2} \right) \) so \( M(2, 1.5) \).

(b) Let D be the point \((a, b)\) then the mid-point of CD is \( \left( \frac{a + 1}{2}, \frac{b + 4}{2} \right) \).

For this to be true \( \frac{a + 1}{2} = 2 \) and \( \frac{b + 4}{2} = \frac{3}{2} \) \( \Rightarrow \) \( a = 3 \) and \( b = -1 \).

The coordinates of D are \((3, -1)\).

EXERCISE 3B

1 Find the coordinates of the mid-point of the line segments joining:

   (a) \((3, 2)\) and \((7, 2)\),  \(\text{ (b) (1, −2)}\) and \((1, 3)\),
   (c) \((0, 3)\) and \((6, 1)\),  \(\text{ (d) (−3, 3)}\) and \((−1, 6)\),
   (e) \((-4, 2)\) and \((3, 6)\),  \(\text{ (f) (−3, −2)}\) and \((−6, 1)\),
   (g) \((-2, 5)\) and \((2, −1)\),  \(\text{ (h) (−2, 5)}\) and \((6, −3)\),
   (i) \((-1.5, 6)\) and \((3.5, 0)\)  \(\text{ (j) (−3.5, 2)}\) and \((4, −1)\).

2 M is the mid-point of the straight line segment PQ. Find the coordinates of Q for each of the cases:

   (a) \(P(2, 2), M(3, 4)\),  \(\text{ (b) } P(2, 1), M(3, 3)\),
   (c) \(P(2, 3), M(1, 5)\),  \(\text{ (d) } P(−2, −5), M(3, 0)\),
   (e) \(P(−2, 4), M(1, 2)\),  \(\text{ (f) } P(−1, −3), M(2, −4)\).

3 The mid-point of \(AB\), where \(A(3, −1)\) and \(B(4, 5)\), is also the mid-point of \(CD\), where \(C(0, 1)\).

   (a) Find the coordinates of D.
   (b) Show that \(AC = BD\).

4 The mid-point of the line segment joining \(A(−1, 3)\) and \(B(5, −1)\) is \(D\). The point \(C\) has coordinates \((4, 4)\). Show that \(CD\) is perpendicular to \(AB\).
3.4 The gradient of a straight line joining two known points

The gradient of a straight line is a measure of how steep it is.

These three lines slope upwards from left to right. They have gradients which are positive. Line (2) is steeper than line (1) so the gradient of (2) is greater than the gradient of (1).

These three lines are all parallel to the x-axis. They are not sloping. Horizontal lines have gradient $0$.

These three lines slope downwards from left to right. They have gradients which are negative.

The gradient of the line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Lines which are equally steep are parallel. Parallel lines have equal gradients.
Worked example 3.6

O(0, 0), P(3, 6), Q(0, 5) and R(−2, 1) are four points.

(a) Find the gradient of the line segment (i) OP, (ii) RQ.

(b) Find the gradient of the line segment (i) OR, (ii) PQ.

(c) What can you deduce from your answers?

Solution

(a) (i) Gradient of OP = \(\frac{6 - 0}{3 - 0} = \frac{6}{3} = 2\);

(ii) Gradient of RQ = \(\frac{5 - 1}{0 - (−2)} = \frac{4}{2} = 2\).

(b) (i) Gradient of OR = \(\frac{1 - 0}{−2 - 0} = \frac{1}{−2} = \frac{1}{2}\);

(ii) Gradient of PQ = \(\frac{6 - 5}{3 - 0} = \frac{1}{3}\).

(c) The lines OP and RQ have gradients which are equal so they are parallel. Lines OR and PQ are not parallel since their gradients are not equal.

So we can deduce that the quadrilateral OPQR is a trapezium.

EXERCISE 3C

1 By finding the gradients of the lines AB and CD determine if the lines are parallel.

(a) A(2, 3) B(3, 5) C(0, 1) D(1, 3),

(b) A(3, 2) B(5, 1) C(−4, −3) D(−2, −2),

(c) A(−4, 5) B(4, 5) C(−1, −2) D(0, −2),

(d) A(−6, −3) B(1, −2) C(2, 0) D(7, 1).

2 By finding the gradients of the lines AB and BC show that A(−2, 3), B(2, 2) and C(6, 1) are collinear points.

3 A(1, −3), B(4, −2) and C(6, 0) are the vertices of triangle ABC.

(a) Find the gradient of each side of the triangle.

(b) Which side of the triangle is parallel to OP where O is the origin and P is the point (1, −11)?
3.5 The gradients of perpendicular lines

Rotate the shaded triangles clockwise through 90° as shown, keeping \( O \) fixed.

Line \( OA \rightarrow \text{line} \ OA' \)
\( A(3, 1) \rightarrow A'(1, -3) \)

Gradient of \( OA = \frac{1}{3} \)
Gradient of \( OA' = \frac{-3}{1} \)

\( OA \) is perpendicular to \( OA' \)

Gradient of \( OA \times \text{Gradient} \ OA' = \frac{1}{3} \times \frac{-3}{1} = -1 \)

In general
gradient of \( OP = \frac{b}{a} \)

Lines are perpendicular if the product of their gradients is \(-1\).

Worked example 3.7
Find the gradient of a line which is perpendicular to the line joining \( A(1, 3) \) and \( B(4, 5) \).
Solution

Gradient of $AB = \frac{5 - 3}{4 - 1} = \frac{2}{3}$.

Let $m_2$ be the gradient of any line perpendicular to $AB$ then

$\frac{2}{3} \times m_2 = -1 \Rightarrow m_2 = -\frac{3}{2}$.

The gradient of any line perpendicular to $AB$ is $-\frac{3}{2}$.

**EXERCISE 3D**

1 Write down the gradient of lines perpendicular to a line with gradient:

(a) $\frac{2}{5}$,  
(b) $-\frac{1}{3}$,  
(c) 4,  
(d) $-\frac{3}{5}$,  
(e) $2\frac{1}{2}$.

2 Two vertices of a rectangle $ABCD$ are $A(-2, 3)$ and $B(4, 1)$. Find:

(a) the gradient of $DC$,  
(b) the gradient of $BC$.

3 $A(1, 2)$, $B(3, 6)$ and $C(7, 4)$ are the three vertices of a triangle.

(a) Show that $ABC$ is a right-angled isosceles triangle.  
(b) $D$ is the point $(5, 0)$. Show that $BD$ is perpendicular to $AC$.  
(c) Explain why $ABCD$ is a square.  
(d) Find the area of $ABCD$.

3.6 The $y = mx + c$ form of the equation of a straight line

Consider any straight line which crosses the $y$-axis at the point $A(0, c)$. We say that $c$ is the $y$-intercept.

Let $P(x, y)$ be any other point on the line.

If the gradient of the line is $m$ then

$$\frac{y - c}{x - 0} = m$$

so $y - c = mx$ or $y = mx + c$.

Let $y = mx + c$ is the equation of a straight line with gradient $m$ and $y$-intercept $c$.

$a x + b y + c = 0$ is the general equation of a line. It has gradient $-\frac{a}{b}$ and $y$-intercept $-\frac{c}{b}$.
Worked example 3.8
A straight line has gradient 2 and crosses the y-axis at the point (0, −4). Write down the equation of the line.

Solution
The line crosses the y-axis at (0, −4) so the y-intercept is −4, so \( c = -4 \). The gradient of the line is 2, so \( m = 2 \).

Using \( y = mx + c \) you get \( y = 2x - 4 \).

The equation of the line is \( y = 2x - 4 \).

Worked example 3.9
The general equation of a straight line is \( Ax + By + C = 0 \).

Find the gradient of the line, and the y-intercept.

Solution
You need to rearrange the equation \( Ax + By + C = 0 \) into the form \( y = mx + c \).

You can write \( Ax + By + C = 0 \) as \( By = -Ax - C \)

or \( y = \frac{-A}{B}x - \frac{C}{B} \)

Compare with \( y = mx + c \),

you see that \( m = \frac{-A}{B} \) and \( c = -\frac{C}{B} \),

so \( Ax + By + C = 0 \) is the equation of a line

with gradient \( \frac{A}{B} \) and y-intercept \( \frac{-C}{B} \).

EXERCISE 3E
1 Find, in the form \( ax + by + c = 0 \), the equation of the line which has:

(a) gradient 2 and y-intercept −3,
(b) gradient \( \frac{2}{3} \) and y-intercept 2,
(c) gradient \( \frac{1}{2} \) and y-intercept −3.

2 Find the gradient and y-intercept for the line with equation:

(a) \( y = 2 + 3x \),
(b) \( 2y = 4x - 5 \),
(c) \( 4y - 7 = 2x \),
(d) \( 2x + 3y = 8 \),
(e) \( 8 - 5x + 4y = 0 \),
(f) \( 0.5y = 4x - 3 \),
(g) \( 5y - 3x = -2 \),
(h) \( 4 - 3x = 2y \),
(i) \( -2.5y + 5x = 3 \),
(j) \( 2y = 4 \).
3.7 The $y - y_1 = m(x - x_1)$ form of the equation of a straight line

Consider any line which passes through the known point $A(x_1, y_1)$ and let $P(x, y)$ be any other point on the line.

If $m$ is the gradient of the line $AP$, then

$$\frac{y - y_1}{x - x_1} = m$$

or $y - y_1 = m(x - x_1)$.

The equation of the straight line which passes through the point $(x_1, y_1)$ and has gradient $m$ is $y - y_1 = m(x - x_1)$.

**Worked example 3.10**

Find the equation of the straight line which is parallel to the line $y = 4x - 1$ and passes through the point $(3, 2)$.

**Solution**

The gradient of the line $y = 4x - 1$ is 4,

so the gradient of any line parallel to $y = 4x - 1$ is also 4.

We need the line with gradient 4 and through the point $(3, 2)$,

so its equation is, using $y - y_1 = m(x - x_1)$,

$$y - 2 = 4(x - 3)$$

or $y = 4x - 10$.

**Worked example 3.11**

(a) Find a Cartesian equation for the perpendicular bisector of the line joining $A(2, 1)$ and $B(4, -5)$.

(b) This perpendicular bisector cuts the coordinate axes at $C$ and $D$. Show that $CD = 1.5 \times AB$. 
Solution
First, we draw a rough sketch.

(a) The gradient of $AB = \frac{-5 - 1}{4 - 2} = -3,$
so the gradient of the perpendicular is $\frac{1}{3}.$

The mid-point of $AB$ is $\left( \frac{2 + 4}{2}, \frac{1 + (-5)}{2} \right) = (3, -2).$

The perpendicular bisector is a straight line which passes through the point $(3, -2)$ and has gradient $\frac{1}{3},$ so its equation is
\[
y - (-2) = \frac{1}{3}(x - 3) \text{ or } y = \frac{1}{3}x - 3.
\]

(b) Let $C$ be the point where the line $y = -\frac{1}{3}x - 3$ cuts the $y$-axis.
When $x = 0, y = -\frac{1}{3} \times 0 - 3 = -3,$ so we have $C(0, -3).$

Let $D$ be the point where the line $y = -\frac{1}{3}x - 3$ cuts the $x$-axis.
When $y = 0, 0 = -\frac{1}{3}x - 3$
\[\Rightarrow x = 9, \text{ so we have } D(9, 0)\]

Distance $CD = \sqrt{(9 - 0)^2 + (0 - (-3))^2} = \sqrt{81 + 9} = \sqrt{90}$
Distance $AB = \sqrt{(4 - 2)^2 + (-5 - 1)^2} = \sqrt{4 + 36} = \sqrt{40}$
\[
\frac{CD}{AB} = \frac{\sqrt{90}}{\sqrt{40}} = \frac{\sqrt{90}}{\sqrt{40}} \cdot \frac{\sqrt{4}}{\sqrt{4}} = \frac{3}{2} = 1.5
\]
so $CD = 1.5 \times AB.$

EXERCISE 3F

1. Find an equation for the straight line with gradient 2 and which passes through the point $(1, 6).$

2. Find a Cartesian equation for the straight line which has gradient $-\frac{1}{3}$ and which passes through $(6, 0).$

3. Find an equation of the straight line passing through $(-1, 2)$ which is parallel to the line with equation $2y = x + 4.$

4. Find an equation of the straight line that is parallel to $3x - 2y - 4 = 0$ and which passes through $(1, 3).$
5 Find an equation of the straight line which passes through the origin and is perpendicular to the line \( y = \frac{1}{2}x + 3 \).

6 Find the \( y \)-intercept of the straight line which passes through the point \((-2, 2)\) and is perpendicular to the line \( 3y = 2x + 1 \).

7 Find a Cartesian equation for the perpendicular bisector of the line joining \( A(2, 3) \) and \( B(0, 6) \).

8 The vertex, \( A \), of a rectangle \( ABCD \), has coordinates \((2, 1)\). The equation of \( BC \) is \( y = \frac{1}{2}x + 3 \). Find, in the form \( y = mx + c \), the equation of:
   (a) \( AD \),
   (b) \( AB \).

9 Given the points \( A(0, 3) \), \( B(5, 4) \), \( C(4, -1) \) and \( E(2, 1) \):
   (a) show that \( BE \) is the perpendicular bisector of \( AC \),
   (b) find the coordinates of the point \( D \) so that \( ABCD \) is a rhombus,
   (c) find an equation for the straight line through \( D \) and \( A \).

10 The perpendicular bisector of the line joining \( A(0, 1) \) and \( C(4, -7) \) intersects the \( x \)-axis at \( B \) and the \( y \)-axis at \( D \). Find the area of the quadrilateral \( ABCD \).

11 Show that the equation of any line parallel to \( ax + by + c = 0 \) is of the form \( ax + by + k = 0 \).

### 3.8 The equation of a straight line passing through two given points

The equation of the straight line which passes through the points \((x_1, y_1)\) and \((x_2, y_2)\) is

\[
\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}
\]

The derivation of this equation is similar to previous ones and is left as an exercise to the reader.

**Worked example 3.12**

Find a Cartesian equation of the straight line which passes through the points \((2, 3)\) and \((-1, 0)\).
Solution

If you take \((x_1, y_1) = (2, 3)\) and \((x_2, y_2) = (-1, 0)\) and substitute into the general equation

\[
\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}
\]

you get

\[
\frac{y - 3}{0 - 3} = \frac{x - 2}{-1 - 2}
\]

or \(y - 3 = x - 2\) which leads to \(y = x + 1\).

3.9 The coordinates of the point of intersection of two lines

In this section you will need to solve simultaneous equations. If you need further practice, see section 1.3 of chapter 3.

A point lies on a line if the coordinates of the point satisfy the equation of the line.

When two lines intersect, the point of intersection lies on both lines. The coordinates of the point of intersection must satisfy the equations of both lines. The equations of the lines must be satisfied simultaneously.

Given accurately drawn graphs of the two intersecting straight lines with equations \(ax + by + c = 0\) and \(Ax + By + C = 0\), the coordinates of the point of intersection can be read off. These coordinates give the solution of the simultaneous equations \(ax + by + c = 0\) and \(Ax + By + C = 0\).

From the diagram, you can clearly read off \((1, 2)\) as the point of intersection of the lines \(x - y + 1 = 0\) and \(2x + y - 4 = 0\). So the solution to the simultaneous equations \(x - y + 1 = 0\) and \(2x + y - 4 = 0\) is \(x = 1, y = 2\).

To find the coordinates of the point of intersection of the two lines with equations \(ax + by + c = 0\) and \(Ax + By + C = 0\), you solve the two linear equations simultaneously.

Note that you can check that your line passes through the points \((2, 3)\) and \((-1, 0)\) by seeing if the points satisfy the equation \(y = x + 1\).

Checking for \((2, 3)\)

LHS = 3    RHS = 2 + 1 = 3  ✔

Checking for \((-1, 0)\)

LHS = 0    RHS = -1 + 1 = 0  ✔
Worked example 3.13
Find the coordinates of the point of intersection of the straight lines with equations \( y = x + 2 \) and \( y = 4x - 1 \).

Solution
At the point of intersection \( P \), \( y = x + 2 \) and \( y = 4x - 1 \).
So eliminating \( y \) gives \( 4x - 1 = x + 2 \).
Rearranging gives \( 4x - x = 2 + 1 \Rightarrow 3x = 3 \Rightarrow x = 1 \).
The \( x \)-coordinate of \( P \) is 1.
To find the \( y \)-coordinate of \( P \), put \( x = 1 \) into \( y = x + 2 \) \( \Rightarrow y = 3 \).
The point of intersection is \((1, 3)\).

Worked example 3.14
The straight lines with equations \( 5x + 3y = 7 \) and \( 3x - 7y = 13 \) intersect at the point \( R \). Find the coordinates of \( R \).

Solution
To find the point of intersection \( R \), you need to solve the simultaneous equations
\[
5x + 3y = 7 \quad [A] \\
3x - 7y = 13 \quad [B] \\
35x + 21y = 49 \quad [C] \\
9x - 21y = 39 \quad [D] \\
44x = 88 \\
\Rightarrow x = 2.
\]
Substituting \( x = 2 \) in \( [A] \) gives \( 10 + 3y = 7 \) \( \Rightarrow 3y = -3 \Rightarrow y = -1 \).
The coordinates of the point \( R \) are \((2, -1)\).

Hint. Checking that the coordinates satisfy each line equation is advisable, especially if the result is being used in later parts of an examination question. It can usually be done in your head rather than on paper.
EXERCISE 3G

1 Verify that (2, 5) lies on the line with equation $y = 3x - 1$.

2 Which of the following points lie on the line with equation $3x + 2y = 6$:
   (a) (3, 0),        (b) (2, 0),        (c) (4, -3)
   (d) (-2, 6),       (e) (0, 2)?

3 Find the coordinates of the points where the following lines intersect the x-axis:
   (a) $y = x + 4$,        (b) $y = 2x - 6$,
   (c) $2x + 3y + 6 = 0$,  (d) $3x - 4y + 12 = 0$.

4 The point $(k, 2k)$ lies on the line with equation $2x + 3y - 6 = 0$. Find the value of $k$.

5 Show that the point $(-4, 8)$ lies on the line passing through the points $(1, 3)$ and $(7, -3)$.

6 (a) Find the equation of the line $AB$ where $A$ is the point $(-3, 7)$ and $B$ is the point $(5, -1)$.
   (b) The point $(k, 3)$ lies on the line $AB$. Find the value of the constant $k$.

7 $A(-5, 2), B(-2, 3), C(-2, -1)$ and $D(-4, -2)$ are the vertices of the quadrilateral $ABCD$.
   (a) Find the equation of the diagonal $BD$.
   (b) Determine whether or not the mid-point of $AC$ lies on the diagonal $BD$.

8 Find the coordinates of the point of intersection of these pairs of straight lines:
   (a) $y = 2x + 7$ and $y = x + 1$,
   (b) $3y + x = 7$ and $2y - x = 3$,
   (c) $5x + 2y = 16$ and $3x + 2y = 8$,
   (d) $y = 8x$ and $y = 40 + 3x$,
   (e) $y = -7$ and $5y = -x - 1$,
   (f) $y = 3x + 3$ and $2y - 5x = 9$,
   (g) $4y + 9x = 8$ and $5y + 6x = 3$,
   (h) $8y = 3x - 11$ and $2x - 5y = 6$.

9 Point $A$ has coordinates $\left(\frac{11}{2}, -1\right)$, point $B$ has coordinates $\left(-3, \frac{61}{60}\right)$ and point $C$ has coordinates $\left(-\frac{19}{6}, \frac{1}{2}\right)$. The straight line $AC$ has equation $12x + 65y - 1 = 0$ and the straight line $BC$ has equation $60y = 150x + 511$. Write down the solution of the simultaneous equations $12x + 65y - 1 = 0$ and $60y = 150x + 511$. 

The straight line $23x + 47y + 105 = 0$ passes through the point of intersection of the two straight lines $y = x$ and $7x - 5y = -3$. Write down the solution of the simultaneous equations $23x + 47y + 105 = 0$ and $7x - 5y = -3$.

**Worked example 3.15**

$ABCD$ is a parallelogram in which the coordinates of $A$, $B$ and $C$ are $(1, 2)$, $(7, -1)$ and $(-1, -2)$, respectively.

(a) Find the equations of $AD$ and $CD$.

(b) Find the coordinates of $D$.

(c) Prove that angle $BAC = 90^\circ$.

(d) Calculate the area of the parallelogram.

(e) Show that the length of the perpendicular from $A$ to $BC$ is $\frac{6\sqrt{65}}{13}$.

**Solution**

(a) $AD$ is parallel to $BC$ and $CD$ is parallel to $BA$.

Gradient of $BC = \frac{-1 - (-2)}{7 - (-1)} = \frac{1}{8}$ so its equation is $y - 2 = \frac{1}{8}(x - 1)$ or $8y - x = 15$.

Gradient of $BA = \frac{-1 - 2}{7 - 1} = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow$ gradient of $CD = -\frac{1}{2}$

$CD$ is a line through $(-1, -2)$ and has gradient $-\frac{1}{2}$ so its equation is $y - (-2) = -\frac{1}{2}[x - (-1)]$ or $2y + x = -5$. 

Always start with a good sketch; this helps to spot obvious errors like wrong signs for gradients or wrong quadrants for points.

Note that when it says the parallelogram $ABCD$, the points must be connected in that order which determines where $D$ must be.

Opposite sides of a parallelogram are equal and parallel.

Using $y - y_1 = m(x - x_1)$.

Using $y - y_1 = m(x - x_1)$.

You could check the signs of the two gradients using your sketch.
(b) \(D\) is the point of intersection of \(AD\) and \(CD\).

Solving \(8y - x = 15\)
and \(2y + x = -5\) simultaneously,
adding gives \(10y + 10 = y = 1\).

Substitution in the second equation gives \(2(1) + x = -5\)
\(\Rightarrow x = -7\), i.e. \(D(-7, 1)\).

(c) Gradient of \(AC = \frac{2 - (-2)}{1 - (-1)} = \frac{4}{2} = 2\).

From earlier work, gradient of \(BA = -\frac{1}{2}\).

Gradient of \(AC \times \) gradient of \(BA = 2 \times -\frac{1}{2} = -1\),
so \(AC\) is perpendicular to \(BA\) and angle \(BAC = 90^\circ\).

(d) Since angle \(BAC = 90^\circ\) you can use:

Area of parallelogram \(ABCD = \text{base} \times \text{height} = AB \times AC\).

\[AB = \sqrt{(7 - 1)^2 + (-1 - 2)^2} = \sqrt{36 + 9} = \sqrt{45}\]
\[AC = \sqrt{(-1 - 1)^2 + (-2 - 2)^2} = \sqrt{4 + 16} = \sqrt{20}\]

So area of parallelogram \(ABCD = \frac{\sqrt{45} \times \sqrt{20}}{2} = \frac{\sqrt{9 \times 100}}{2} = 30\) square units

(e) Using the base of the parallelogram as \(CB\) and letting \(h = \text{length of the perpendicular from } A\) to \(BC\),

\[\Rightarrow \text{area of parallelogram } ABCD = CB \times h = 30 = \sqrt{(7 - 1)^2 + (-1 - (-2))^2} \times h\]
\[30 = \sqrt{64 + 1} \times h \Rightarrow h = \frac{30}{\sqrt{65}} = \frac{30\sqrt{65}}{65} = \frac{6\sqrt{65}}{13}\]

MIXED EXERCISE

1. The point \(A\) has coordinates \((2, 3)\) and \(O\) is the origin.

(a) Write down the gradient of \(OA\) and hence find the equation of the line \(OA\).

(b) Show that the line which has equation \(4x + 6y = 13\):

(i) is perpendicular to \(OA\),
(ii) passes through the mid-point of \(OA\). [A]
2 The line $AB$ has equation $5x - 2y = 7$. The point $A$ has coordinates $(1, -1)$ and the point $B$ has coordinates $(3, k)$.

(a) (i) Find the value of $k$.

(ii) Find the gradient of $AB$.

(b) Find an equation for the line through $A$ which is perpendicular to $AB$.

(c) The point $C$ has coordinates $(-6, -2)$. Show that $AC$ has length $p\sqrt{2}$, stating the value of $p$. [A]

3 The point $P$ has coordinates $(1, 10)$ and the point $Q$ has coordinates $(4, 4)$.

(a) Show that the length of $PQ$ is $3\sqrt{5}$.

(b) (i) Find the equation of the perpendicular bisector of $PQ$.

(ii) This perpendicular bisector intersects the $x$-axis at the point $A$. Find the coordinates of $A$.

4 The point $A$ has coordinates $(3, 5)$ and the point $B$ has coordinates $(1, 1)$.

(a) (i) Find the gradient of $AB$.

(ii) Show that the equation of the line $AB$ can be written in the form $rx + sy = s$, where $r$ and $s$ are positive integers.

(b) The mid-point of $AB$ is $M$ and the line $MC$ is perpendicular to $AB$.

(i) Find the coordinates of $M$.

(ii) Find the gradient of the line $MC$.

(iii) Given that $C$ has coordinates $(5, p)$, find the value of the constant $p$. [A]

5 The points $A$ and $B$ have coordinates $(13, 5)$ and $(9, 2)$, respectively.

(a) (i) Find the gradient of $AB$.

(ii) Find an equation for the line $AB$.

(b) The point $C$ has coordinates $(2, 3)$ and the point $X$ lies on $AB$ so that $XC$ is perpendicular to $AB$.

(i) Show that the equation of the line $XC$ can be written in the form $4x + 3y = 17$.

(ii) Calculate the coordinates of $X$. [A]

6 The equation of the line $AB$ is $5x - 3y = 26$.

(a) Find the gradient of $AB$.

(b) The point $A$ has coordinates $(4, -2)$ and a point $C$ has coordinates $(-6, 4)$.

(i) Prove that $AC$ is perpendicular to $AB$.

(ii) Find an equation for the line $AC$, expressing your answer in the form $px + qy = r$, where $p, q$ and $r$ are integers.

(c) The line with equation $x + 2y = 13$ also passes through the point $B$. Find the coordinates of $B$. [A]
7 The points $A$, $B$ and $C$ have coordinates $(1, 7)$, $(5, 5)$ and $(7, 9)$, respectively.

(a) Show that $AB$ and $BC$ are perpendicular.

(b) Find an equation for the line $BC$.

(c) The equation of the line $AC$ is $3y = x + 20$ and $M$ is the mid-point of $AB$.

(i) Find an equation of the line through $M$ parallel to $AC$.

(ii) This line intersects $BC$ at the point $T$. Find the coordinates of $T$.

8 The point $A$ has coordinates $(3, 5)$, $B$ is the point $(-5, 1)$ and $O$ is the origin.

(a) Find, in the form $y = mx + c$, the equation of the perpendicular bisector of the line segment $AB$.

(b) This perpendicular bisector cuts the $y$-axis at $P$ and the $x$-axis at $Q$.

(i) Show that the line segment $BP$ is parallel to the $x$-axis.

(ii) Find the area of triangle $OPQ$.

9 The points $A (-1, 2)$ and $C (5, 1)$ are opposite vertices of a parallelogram $ABCD$. The vertex $B$ lies on the line $2x + y = 5$. The side $AB$ is parallel to the line $3x + 4y = 8$. Find:

(a) the equation of the side $AB$,

(b) the coordinates of $B$,

(c) the equations of the sides $AD$ and $CD$,

(d) the coordinates of $D$.

10 $ABCD$ is a rectangle in which the coordinates of $A$ and $C$ are $(0, 4)$ and $(11, 1)$, respectively, and the gradient of the side $AB$ is $-5$.

(a) Find the equations of the sides $AB$ and $BC$.

(b) Show that the coordinates of $B$ is $(1, -1)$.

(c) Calculate the area of the rectangle.

(d) Find the coordinates of the point on the $y$-axis which is equidistant from points $A$ and $D$.

[A]
The distance between the points \((x_1, y_1)\) and \((x_2, y_2)\) is
\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

The coordinates of the mid-point of the line segment joining \((x_1, y_1)\) and \((x_2, y_2)\) are
\[
\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).
\]

The gradient of a line joining the two points \(A(x_1, y_1)\) and \(B(x_2, y_2)\) is
\[
\frac{y_2 - y_1}{x_2 - x_1}.
\]

Lines with gradients \(m_1\) and \(m_2\):
- are parallel if \(m_1 = m_2\),
- are perpendicular if \(m_1 \times m_2 = -1\).

The equation of the straight line which passes through the point \((x_1, y_1)\) and has gradient \(m\) is
\[
y - y_1 = m(x - x_1).
\]

The equation of the straight line which passes through the points \((x_1, y_1)\) and \((x_2, y_2)\) is
\[
\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.
\]

A point lies on a line if the coordinates of the point satisfy the equation of the line.

Given accurately drawn graphs of the two intersecting straight lines with equations \(ax + by + c = 0\) and \(Ax + By + C = 0\), the coordinates of the point of intersection can be read off. These coordinates give the solution of the simultaneous equations \(ax + by + c = 0\) and \(Ax + By + C = 0\).

To find the coordinates of the point of intersection of the two lines with equations \(ax + by + c = 0\) and \(Ax + By + C = 0\), you solve the two equations simultaneously.

**Key point summary**

1. The distance between the points \((x_1, y_1)\) and \((x_2, y_2)\) is \(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\).
2. The coordinates of the mid-point of the line segment joining \((x_1, y_1)\) and \((x_2, y_2)\) are \(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\).
3. The gradient of a line joining the two points \(A(x_1, y_1)\) and \(B(x_2, y_2)\) is \(\frac{y_2 - y_1}{x_2 - x_1}\).
4. Lines with gradients \(m_1\) and \(m_2\):
   - are parallel if \(m_1 = m_2\),
   - are perpendicular if \(m_1 \times m_2 = -1\).
5. \(y = mx + c\) is the equation of a straight line with gradient \(m\) and \(y\)-intercept \(c\).
6. \(ax + by + c = 0\) is the general equation of a line.
   It has gradient \(-\frac{a}{b}\) and \(y\)-intercept \(-\frac{c}{b}\).
7. The equation of the straight line which passes through the point \((x_1, y_1)\) and has gradient \(m\) is \(y - y_1 = m(x - x_1)\).
8. The equation of the straight line which passes through the points \((x_1, y_1)\) and \((x_2, y_2)\) is \(\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}\).
9. A point lies on a line if the coordinates of the point satisfy the equation of the line.
10. Given accurately drawn graphs of the two intersecting straight lines with equations \(ax + by + c = 0\) and \(Ax + By + C = 0\), the coordinates of the point of intersection can be read off. These coordinates give the solution of the simultaneous equations \(ax + by + c = 0\) and \(Ax + By + C = 0\).
11. To find the coordinates of the point of intersection of the two lines with equations \(ax + by + c = 0\) and \(Ax + By + C = 0\), you solve the two equations simultaneously.
Test yourself

1 Calculate the distance between the points (2, -3) and (7, 9).

2 State the coordinates of the mid-point of the line segment PQ where P(3, -2) and Q(7, 1).

3 Find the gradient of the line joining the points A and B where A is the point (-3, -2) and B is the point (-5, 4).

4 The lines CD and EF are perpendicular with points C(1, 2), D(3, -4), E(-2, 5) and F(k, 4). Find the value of the constant k.

5 Find a Cartesian equation of the line which passes through the point (-2, 1) and is perpendicular to the line 5y + 3x = 7.

6 Find the point of intersection of the lines with equations 3x - 5y = 11 and y = 4x - 9.

What to review

Section 3.2

Section 3.3

Section 3.4

Section 3.5

Sections 3.5 and 3.7

Section 3.9

Questions:

1. \((1 - t^2)\, 9\)
2. \(\sqrt{1 + 3s - 4t}\, 8\)
3. \(s - \pi\, 8\)
4. \(s - \pi\, 8\)
5. \(t - 8\, 8\)
6. \(t\)