CHAPTER 2

C3: Transformations of graphs and the modulus function

Learning objectives

After studying this chapter, you should be able to:

- transform simple graphs to produce other graphs
- understand the effect of composite transformations on equations of curves and describe them geometrically
- understand what is meant by a modulus function
- sketch graphs of functions involving modulus functions
- solve equations and inequalities involving modulus functions.

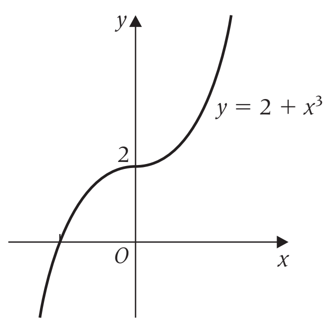
2.1 Review of simple transformations of graphs

Although you may use a graphics calculator to draw graphs, it is important to see how the graph of one curve can be obtained from the graph of a simpler curve using a sequence of transformations.

Some simple transformations of graphs were introduced in chapter 5 of C2. The basic results are reviewed below. For instance, the graph of $y = x^2$ can be transformed into the graph of $y = (x - 3)^2 + 4$ by a translation of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

> In general, a translation of $\begin{bmatrix} a \\ b \end{bmatrix}$ transforms the graph of $y = \mathrm{f}(x)$ into the graph of $y = \mathrm{f}(x - a) + b$.

You learnt how to find the equations of new curves after a reflection in one of the coordinate axes.
For example, the graph of $y = 2 + x^3$ is sketched opposite.



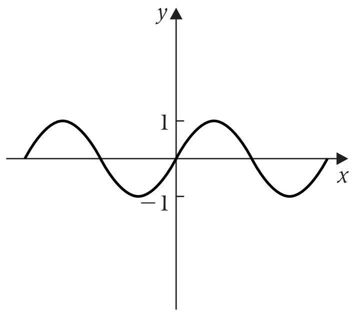
After reflection in the x-axis, the new curve will have equation $y = -2 - x^3$. The general result is given below.

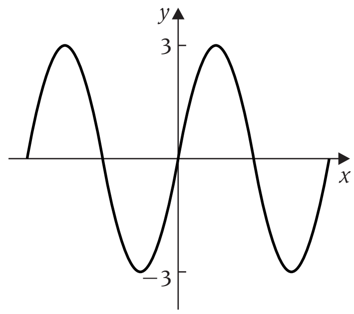
> The graph of $y = \mathrm{f}(x)$ is transformed into the graph of $y = -\mathrm{f}(x)$ by a reflection in the line $y = 0$ (the x-axis).

When the curve $y = 2 + x^3$ is reflected in the y-axis, the new curve has equation $y = 2 - x^3$.

> The graph of $y = \mathrm{f}(x)$ is transformed into the graph of $y = \mathrm{f}(-x)$ by a reflection in the line $x = 0$ (the y-axis).

The graph of $y = \sin x$ is transformed into the graph of $y = 3\sin x$ by a stretch of scale factor 3 in the y-direction.

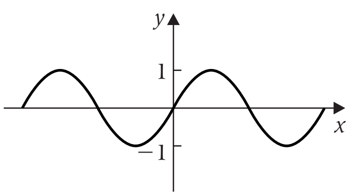


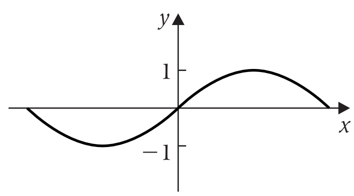


The general result is:

> The graph of $y = \mathrm{f}(x)$ is transformed into the graph of $y = d\,\mathrm{f}(x)$ by a stretch of scale factor d in the y-direction.

A stretch of scale factor 2 in the x-direction transforms the graph of $y = \sin x$ into the graph with equation $y = \sin\left(\frac{x}{2}\right)$.





> The graph of $y = \mathrm{f}(x)$ is transformed into the graph of $y = \mathrm{f}\left(\frac{x}{c}\right)$ by a stretch of scale factor c in the x-direction.

Worked example 2.1

Find the equation of the resulting curve when the curve $y = 2 + \tan x$ is transformed by:

(a) a reflection in the y-axis,

(b) a translation of $\begin{bmatrix} \frac{\pi}{3} \\ 5 \end{bmatrix}$,

(c) a stretch of scale factor 0.5 in the x-direction.

You are transforming the original curve in each case. Successive transformations will be discussed in the next section.

Solution

(a) The graph of $y = \mathrm{f}(x)$ is transformed into the graph of $y = \mathrm{f}(-x)$ by a reflection in the y-axis.
Hence the new curve has equation $y = 2 + \tan(-x)$.
However, since $\tan(-x) = -\tan x$, the equation of the new curve can be written as $y = 2 - \tan x$.

(b) Recall that a translation of $\begin{bmatrix} a \\ b \end{bmatrix}$ transforms the graph of $y = \mathrm{f}(x)$ into the graph of $y = \mathrm{f}(x - a) + b$.

After translation through the vector $\begin{bmatrix} \frac{\pi}{3} \\ 5 \end{bmatrix}$, the curve

$y = 2 + \tan x$ has equation $y = 2 + \tan\left(x - \dfrac{\pi}{3}\right) + 5$

or $y = 7 + \tan\left(x - \dfrac{\pi}{3}\right)$.

(c) The graph of $y = \mathrm{f}(x)$ is transformed into the graph of $y = \mathrm{f}\left(\dfrac{x}{c}\right)$ by a stretch of scale factor c in the x-direction.
Hence $y = 2 + \tan x$ is transformed into $y = 2 + \tan\left(\dfrac{x}{0.5}\right)$
or $y = 2 + \tan 2x$.

Worked example 2.2

Describe geometrically how the first curve is transformed into the second curve in each of the following cases:

(a) $y = x^2$, $\quad y = (x + 3)^2 + 2$

(b) $y = \cos x$, $\quad y = \cos 3x$

(c) $y = 2^x$, $\quad y = 2^{-x}$

Solution

(a) The first curve has been translated through the vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ to give the second curve.

(b) A one-way stretch in the x-direction has taken place. The scale factor is $\frac{1}{3}$.

Notice that x has been divided by $\frac{1}{3}$ to give $3x$.

(c) The curve has been reflected in the y-axis (or $x = 0$).

EXERCISE 2A

1 Find the equation of the resulting curve after each of the following has been translated through $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(a) $y = x^3$ **(b)** $y = x^2 + 4x + 5$

(c) $y = \tan x$ **(d)** $y = 3^x$

2 Find the equation of the resulting curve after each of the following has been stretched in the x-direction by scale factor $\frac{1}{2}$.

(a) $y = x^3$ **(b)** $y = x^2 + 4x + 5$

(c) $y = \tan x$ **(d)** $y = 3^x$

3 Describe geometrically how the curve $y = 1 + \sin x$ is transformed into the following curves:

(a) $y = \sin x$ **(b)** $y = 1 - \sin x$

(c) $y = 5 + \sin (x + 2)$ **(d)** $y = 4 + 4 \sin x$

(e) $y = -1 - \sin x$ **(f)** $y = 1 + \sin 5x$

4 Describe geometrically how the curve $y = 2^x$ is transformed into the following curves:

(a) $y = 2^{5x}$ **(b)** $y = 2^{x-3}$ **(c)** $y = 2^{\frac{x}{3}}$

(d) $y = 2^{x+7}$ **(e)** $y = -2^x$

5 Describe a geometrical transformation which maps the graph of $y = 3^x$ onto:

(a) $y = 3^{-x}$ **(b)** $y = 3^{x-4}$ **(c)** $y = 3^{\frac{5x}{2}}$

(d) $y = 2 \times 3^x$ **(e)** $y = 4 + 3^x$ **(f)** $y = 9^x$

2.2 Composite transformations

You can perform each of the transformations described above in sequence so as to produce a composite transformation.

Worked example 2.3

Find the equation of the resulting curve when the following transformations are performed in sequence on the curve with equation $y = x^3$:

(a) a stretch by a factor 2 in the y-direction,

(b) a translation through $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$,

(c) a reflection in the x-axis.

Solution

(a) After a stretch by a factor 2 in the y-direction, the curve $y = x^3$ becomes the curve $y = 2x^3$.

(b) Applying a translation of $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ to the curve $y = 2x^3$ gives the new curve $y = 2(x + 1)^3 - 3$.

(c) Finally the effect of a reflection in the x-axis is to transform $y = \mathrm{f}(x)$ into the curve $y = -\mathrm{f}(x)$.

The curve $y = 2(x + 1)^3 - 3$ therefore becomes the curve with equation $y = 3 - 2(x + 1)^3$.

It is important to perform the transformations in the given order or you will not obtain the correct final equation.

Worked example 2.4

Describe geometrically a sequence of transformations that transforms $y = \sin x$ into $y = 2 + 5 \sin 3x$.

Solution

The curve $y = \sin x$ is stretched by scale factor $\frac{1}{3}$ in the x-direction to give the curve $y = \sin 3x$.

The two stretches could have been done in any order to map $y = \sin x$ onto $y = 5 \sin 3x$.

Next the curve $y = \sin 3x$ is stretched by scale factor 5 in the y-direction to give $y = 5 \sin 3x$.

Finally $y = 5 \sin 3x$ is translated through $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ to give the final curve with equation $y = 2 + 5 \sin 3x$.

EXERCISE 2B

1 Describe a sequence of transformations that would map the graph of $y = x^3$ onto the graph of $y = 3(x - 5)^3$.

2 Find the resulting curve when the following transformations are applied in sequence to the curve $y = \cos x$.

(a) a reflection in the x-axis,

(b) a translation through $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$,

(c) a stretch by a scale factor 3 in the x-direction.

3 Find the equation of the resulting graph when the graph of $y = 3^x$ is translated through $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ then reflected in the y-axis.

4 Find the equation of the new curve when the $y = \sin x$ is transformed by the following sequence of transformations:

(a) a translation through $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$,

(b) a reflection in the x-axis,

(c) a stretch by a scale factor 4 in the y-direction.

5 Express $2x^2 - 12x + 19$ in the form $2(x - a)^2 + b$. Hence decribe geometrically how the graph of $y = x^2$ can be transformed into the graph of $y = 2x^2 - 12x + 19$.

6 Describe geometrically how the first curve can be transformed into the second curve by a sequence of transformations:

(a) $y = x^2$, $y = 4(x - 2)^2$ **(b)** $y = x^2$, $y = 4 + 3(x + 1)^2$

(c) $y = x^3$, $y = (2x - 1)^3$ **(d)** $y = x^3$, $y = -(x - 3)^3$

(e) $y = x^4$, $y = (3x + 5)^4$ **(f)** $y = x^5$, $y = 4\left(\frac{x}{3} - 2\right)^5$

7 Describe geometrically how the curve $y = 3x^2 - 5$ can be transformed into the curve $y = x^2$ by a sequence of transformations.

8 Describe geometrically how the curve $y = 5\sin(x - 3)$ can be transformed into the curve $y = \sin(x + 1)$ by a sequence of transformations.

9 Describe geometrically how the curve $y = 3 + \cos 2x$ can be transformed into the curve $y = 5\cos x$ by a sequence of transformations.

10 Describe geometrically how the curve $y = 4^{x+3}$ can be transformed into the curve $y = \frac{2^x}{5}$ by a sequence of stretches.

11 **(a)** Describe the geometrical transformation that transforms the graph of $y = 3x$ into the graph of $3y = x$.

(b) Find the equations of the new graphs after each of the folowing has been reflected in the line $y = x$:
(i) $y = 3x + 2$, **(ii)** $y = x^2$, **(iii)** $(x - 1)^2 + y^2 = 4$.

12 The graph of $y = f(x)$ is reflected in the x-axis and then the y-axis to produce the graph with equation $y = g(x)$.

(a) Find $g(x)$ in terms of f and x.

(b) Describe geometrically the single transformation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

13 **(a)** Find the new equation resulting from reflecting the curve $y = x^2$ in the line $y = 1$.

(b) Describe a sequence of simple geometrical transformations that will map $y = x^2$ onto your answer to **(a)**.

14 **(a)** Find the new equation resulting from reflecting the curve $y = 2^x$ in the line $x = 5$.

(b) Describe a sequence of simple geometrical transformations that will map $y = 2^x$ onto your answer to **(a)**.

15 **(a)** Describe a sequence of geometrical transformations that will transform the graph of $y = \mathrm{f}(x)$ into the graph of $y = 6 - \mathrm{f}(x)$.

(b) Describe a single geometrical transformation that will transform the graph of $y = \mathrm{f}(x)$ into the graph of $y = 6 - \mathrm{f}(x)$.

(c) Describe a single geometrical transformation that will transform the graph of $y = \mathrm{f}(x)$ into the graph of $y = 2p - \mathrm{f}(x)$.

16 **(a)** Describe a sequence of geometrical transformations that will transform the graph of $y = \mathrm{f}(x)$ into the graph of $y = \mathrm{f}(4 - x)$.

(b) Describe a single geometrical transformation that will transform the graph of $y = \mathrm{f}(x)$ into the graph of $y = \mathrm{f}(4 - x)$.

(c) Describe a single geometrical transformation that will transform the graph of $y = \mathrm{f}(x)$ into the graph of $y = \mathrm{f}(2q - x)$.

2.3 Modulus function

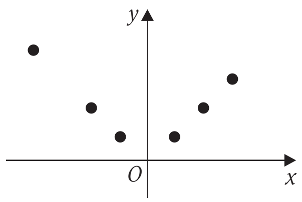
The modulus function finds the absolute value of a number. Any negative sign in front of a number is disregarded and a positive answer is returned. Consider the function box below.

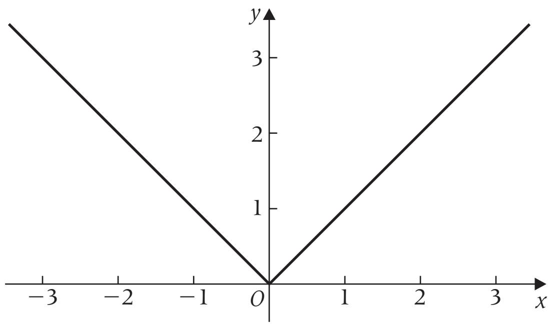
When the value 3 is input then the output is 3, whereas when -7 is input the output is 7. An input of zero gives an output of zero.

The modulus of x is written as $|x|$ and is usually read as 'mod x'.

By taking a set of values of x it is easy to see that the graph of $y = |x|$ would have the appearance of the V shape below.

Often the key on a calculator which finds the modulus is denoted by ABS since it finds the absolute value of a number.





The **modulus function** is actually defined as follows:

$$|x| = \begin{cases} x & \text{when } x \geqslant 0 \\ -x & \text{when } x < 0 \end{cases}$$

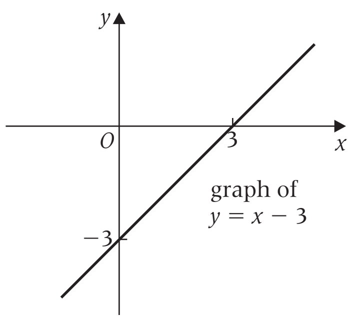
Worked example 2.5

Sketch the graph of $y = |x - 3|$.

Solution

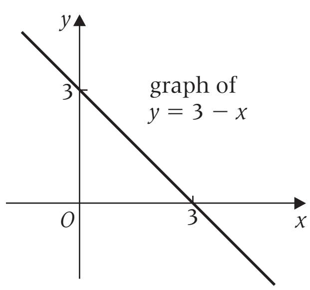
Method 1

When $x \geqslant 3$, $|x - 3| = x - 3$ since $x - 3 \geqslant 0$.



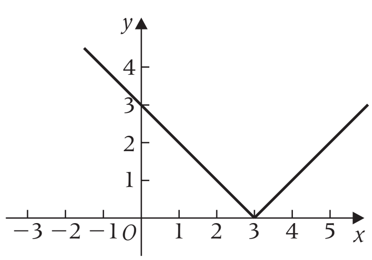
So the graph for $x \geqslant 3$ is the same as the graph of $y = x - 3$.

However when $x < 3$, $x - 3$ is negative and so $|x - 3| = -(x - 3)$.



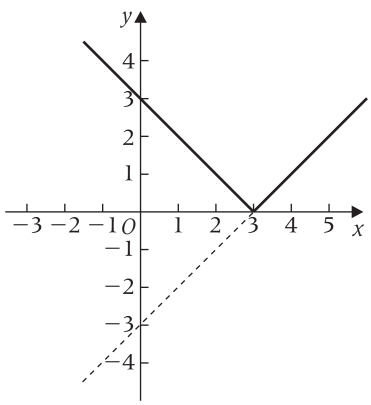
This means that when $x < 3$, the graph of $y = |x - 3|$ is the same as the graph of $y = -(x - 3) = 3 - x$.

The graph of $y = |x - 3|$ is therefore as drawn below.



Method 2

An alternative approach is to draw the graph of $y = x - 3$ and then to reflect the section of the graph that lies below the x-axis in the x-axis.



Worked example 2.6

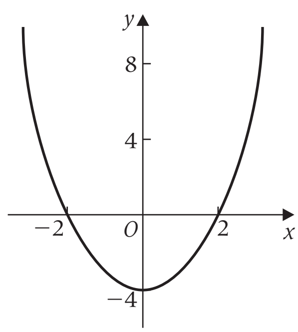
Sketch the graphs of

(a) $y = |x^2 - 4|$ for $-3 \leqslant x \leqslant 3$,

(b) $y = |\sin x|$ for $0 \leqslant x \leqslant 2\pi$.

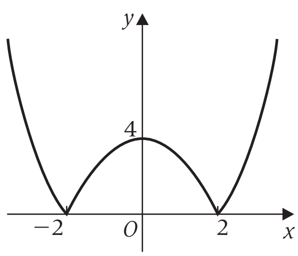
Solution

(a) Draw the graph of $y = x^2 - 4$.



Now reflect in the x-axis all the parts of the graph which lie below the x-axis.

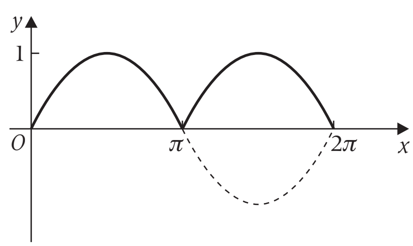
The resulting graph is that of $y = |x^2 - 4|$.



There are now some ‘sharp’ corners on the graph called **cusps**. Do not be tempted to smooth these out.

(b) Draw the graph of $y = \sin x$ for $0 \leqslant x \leqslant 2\pi$.

Reflect in the x-axis the portion of the graph where y is negative so as to produce the graph of $y = |\sin x|$.



Worked example 2.7

Sketch the graphs of:

(a) $y = 3 - |x|$, **(b)** $y = |x - 1| + |x - 2|$.

Solution

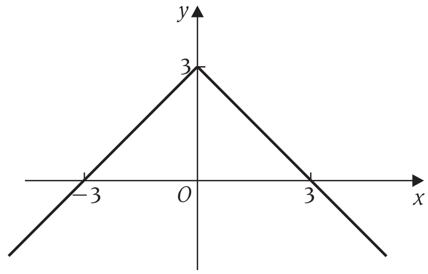
(a) For $x > 0$, the graph is identical to that of $y = 3 - x$.

When $x = 0$, $y = 3$.

For $x < 0$, the graph is identical to that of $y = 3 - (-x) = 3 + x$.

Hence we can sketch the graph of $y = 3 - |x|$.

Alternatively, the graph of $y = 3 - |x|$ can be obtained from the graph of $y = |x|$ by a reflection in the x-axis followed by a translation of $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.



(b) It is necessary to consider three separate intervals in order to sketch $y = |x - 1| + |x - 2|$.

Firstly, when $x < 1$, $|x - 1| = -(x - 1) = -x + 1$ and also $|x - 2| = -(x - 2) = -x + 2$.
Therefore $y = |x - 1| + |x - 2| = -x + 1 + (-x + 2) = 3 - 2x$.
The graph is identical to $y = 3 - 2x$ for $x < 1$.

Next, when $1 < x < 2$, $|x - 1| = x - 1$.
But $|x - 2| = -(x - 2) = -x + 2$.
Therefore $y = |x - 1| + |x - 2| = x - 1 + (-x + 2) = 1$.
The graph is identical to $y = 1$ for $1 < x < 2$.

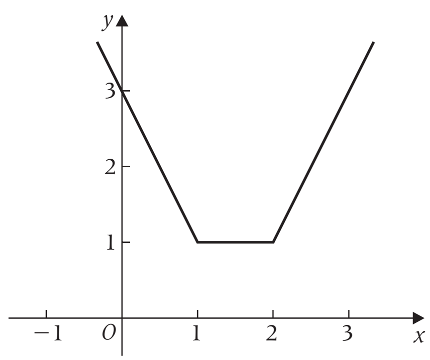
Finally, when $x > 2$ $|x - 1| = x - 1$ and also $|x - 2| = x - 2$.
Therefore $y = |x - 1| + |x - 2| = x - 1 + x - 2 = 2x - 3$.
The graph is identical to $y = 2x - 3$ for $x > 2$.

The graph of $y = |x - 1| + |x - 2|$ can now be sketched.

The end points of these intervals were not included but serve as useful check points.

When $x = 1$, $y = 0 + 1 = 1$.

When $x = 2$, $y = 1 + 0 = 1$.



EXERCISE 2C

Sketch the graph of each of the following, showing the values of any intercepts on the axes.

1 $y = |3x|$ **2** $y = |x + 4|$ **3** $y = |3x - 5|$

4 $y = |5 - 2x|$ **5** $y = |x| - 5$ **6** $y = 4 - |x|$

7 $y = |x^2 - 1|$ **8** $y = |(x - 5)(x + 2)|$

9 $y = |x| + |x - 3|$ **10** $y = |x| - |x - 3|$

11 $y = |(x - 1)(x - 2)(x - 3)|$ **12** $y = |x^2 - 5| + 4$

13 $y = |x| + |x - 1| + |x - 2|$ **14** $y = \sin|x|, -2\pi \leqslant x \leqslant 2\pi$

15 $y = |\cos x|, -2\pi \leqslant x \leqslant 2\pi$ **16** $y = |\tan x|, 0 < x < \pi$

17 $y = |\cos 3x|, -\pi \leqslant x \leqslant \pi$ **18** $y = 1 + |\sin 2x|, -\pi \leqslant x \leqslant \pi$

2.4 Equations involving modulus functions

Often the graphical approach is best since you can see the approximate solutions and how many solutions to expect.

Worked example 2.8

Solve the equation $|3 - 2x| = x - 1$.

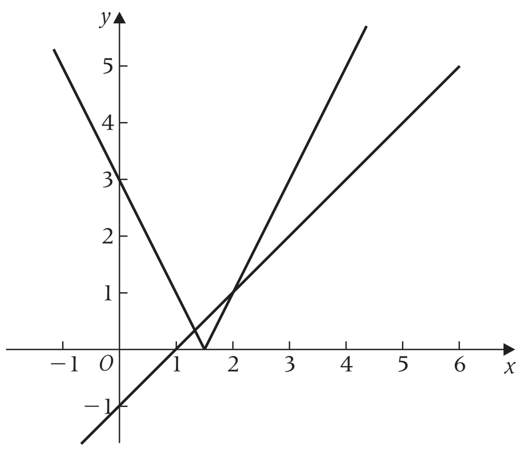
Solution

Method 1

Firstly, sketch the graph of $y = |3 - 2x|$. This is the V-shaped graph. The sections have gradients ± 2.

Now add the straight line $y = x - 1$, which has gradient 1. Hence there are two points of intersection.

This means the equation $|3 - 2x| = x - 1$ will have two solutions, and from the graph one of these is less than 1.5 and the other is greater than 1.5.



When $x < \frac{3}{2}$, $|3 - 2x| = 3 - 2x$.

One solution is given by $3 - 2x = x - 1$

$$\Rightarrow 3 + 1 = 2x + x \Rightarrow x = \frac{4}{3}.$$

This solution is valid since $x < \frac{3}{2}$.

Similarly, when $x > \frac{3}{2}$, $|3 - 2x| = -(3 - 2x)$.

The second solution is given by $-(3 - 2x) = x - 1$

$$\Rightarrow -3 + 2x = x - 1 \Rightarrow 2x - x = 3 - 1 \Rightarrow x = 2.$$

This solution is valid since $x > \frac{3}{2}$.

The two solutions are $x = 2, \frac{4}{3}$.

Method 2

Square both sides of the equation $|3 - 2x| = x - 1$.

$(|3 - 2x|)^2 = (x - 1)^2 \Rightarrow 9 - 12x + 4x^2 = x^2 - 2x + 1$.

$\Rightarrow 3x^2 - 10x + 8 = 0 \quad \Rightarrow (3x - 4)(x - 2) = 0$.

The two solutions are $x = 2, \frac{4}{3}$.

It is always a little dangerous when you square both sides of an equation, since you may produce spurious answers. You should therefore check that these solutions satisfy the original equation. In this case the solutions are valid.

Worked example 2.9

Solve the equation $|x - 2| = -3$.

Solution

An approach involving squaring both sides yields

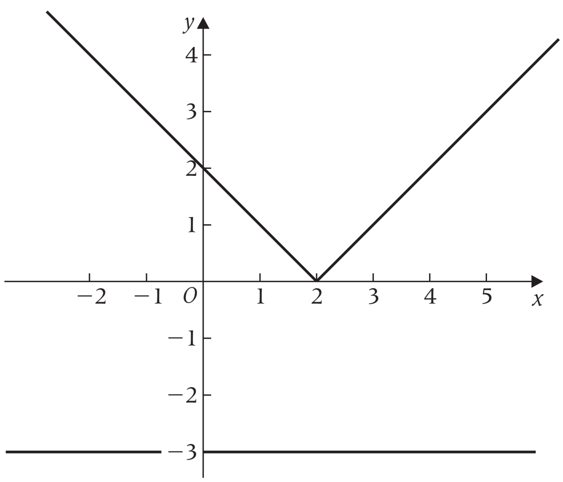
$(|x - 2|)^2 = (-3)^2 \quad \Rightarrow x^2 - 4x + 4 = 9 \Rightarrow x^2 - 4x - 5 = 0$.

$\Rightarrow (x - 5)(x + 1) = 0 \Rightarrow x = 5, x = -1$.

Checking each of these values in the original equation shows that neither of the answers are correct solutions.

When $x = 5$, $|x - 2| = |5 - 2| = 3$.

When $x = -1$, $|x - 2| = |(-1)-2| = |-3| = 3$.



A sketch can easily reveal when no solutions actually exist since the graphs of $y = |x - 2|$ and $y = -3$ do not intersect.

The graph of $y = |x - 2|$ is never negative and so the equation $|x - 2| = -3$ has no solutions.

Can you spot the flaw in the following argument?

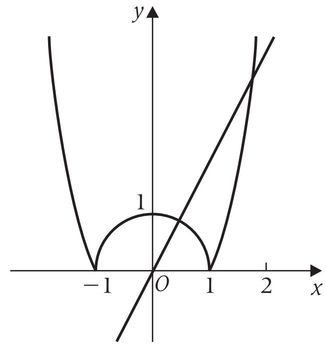
For solutions to $|x - 2| = -3$, solve $x - 2 = -3 \Rightarrow x = 2 - 3 = -1$.

Also solve $-(x - 2) = -3 \Rightarrow -x + 2 = -3 \Rightarrow x = 5$.

Hence the solutions are $x = -1$ and $x = 5$.

Worked example 2.10

Solve the equation $|x^2 - 1| = 6x$.



Solution

A sketch showing the graphs of $y = |x^2 - 1|$ and $y = 6x$ enables you to see that there are two solutions, one in the interval $0 < x < 1$ and the other satisfying $x > 1$.

For $0 < x < 1$, you can write $|x^2 - 1| = -(x^2 - 1) = -x^2 + 1$
Hence, solving

$$-x^2 + 1 = 6x \Rightarrow 0 = x^2 + 6x - 1 \Rightarrow x = \frac{-6 \pm \sqrt{36 + 4}}{2}$$

$$\Rightarrow \quad x = -3 \pm \frac{\sqrt{40}}{2} = -3 \pm \sqrt{10}.$$

Using the quadratic equation formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

But the only valid solution in the interval $0 < x < 1$ is $x = -3 + \sqrt{10} \approx 0.162\ldots$. You must reject the solution that is negative.

For $x > 1$, you can write $|x^2 - 1| = x^2 - 1$

Hence, solving $x^2 - 1 = 6x \Rightarrow 0 = x^2 - 6x - 1 \Rightarrow x = \dfrac{6 \pm \sqrt{36 + 4}}{2}$

$\Rightarrow x = 3 \pm \dfrac{\sqrt{40}}{2} = 3 \pm \sqrt{10}$. Once again, rejecting the negative solution gives the solution $x = 3 + \sqrt{10} \approx 6.162\ldots$.

The two solutions are $x = -3 + \sqrt{10} \approx 0.162\ldots$
$x = 3 + \sqrt{10} \approx 6.162\ldots$.

EXERCISE 2D

Solve the following equations. In some cases there are no solutions.

1 $|x - 3| = 2$ **2** $|4 - x| = 5$ **3** $|x + 2| = 7$

4 $|x - 1| = x$ **5** $|2 - x| = x + 1$ **6** $|2x + 3| = x - 1$

7 $|2x - 3| = -2$ **8** $|4 - 3x| = x$ **9** $|x^2 + 2| = 3x$

10 $|2x^2 - 3x| = -1$ **11** $|4 - 3x^2| = x$

12 $|x^2 + 2x| = 3x + 2$ **13** $|x^2 + x| + 2 = 0$

14 $|4x^2 - x| = -3$ **15** $|4x^2 - 3| = x$

2.5 Inequalities involving modulus functions

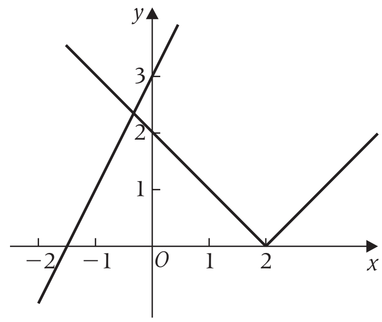
It is advisable to draw a sketch and then to find the critical points when trying to solve inequalities involving modulus functions.

2

Worked example 2.11

Solve the inequality $|x - 2| < 2x + 3$.

Solution



The graphs $y = |x - 2|$ and $y = 2x + 3$ are drawn opposite and there is a single point of intersection when x is negative.

Since $x < 2$ for this point of intersection, you can write

$$|x - 2| = -(x - 2) = 2 - x.$$

Solving $2 - x = 2x + 3$ gives $-1 = 3x$. Hence $x = -\frac{1}{3}$.

This is the critical point and by looking at the graph, the y-value for the graph of $y = |x - 2|$ is less than the y-value on the graph of $y = 2x + 3$ whenever you are to the right-hand side of this critical point.

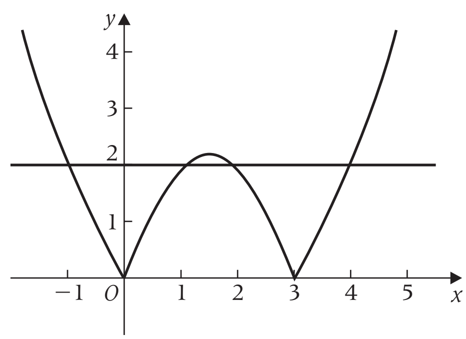
The solution is therefore $x > -\frac{1}{3}$.

You can check this solution by taking a test value. For example when $x = 0$, $|x - 2| = 2$ and $2x + 3 = 3$. Since $2 < 3$, it gives a check on the solution.

Worked example 2.12

Solve the inequality $|x^2 - 3x| > 2$.

Solution



The graph of $y = x^2 - 3x = x(x - 3)$ is a parabola cutting the x-axis at (0, 0) and (3, 0). The graph of $y = |x^2 - 3x|$ is sketched opposite together with the straight line $y = 2$.

There are four points of intersection and hence four critical points.

Two of these are given by solving
$x^2 - 3x = 2 \Rightarrow x^2 - 3x - 2 = 0$.
This equation must be solved using the formula
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Hence $x = \dfrac{3 \pm \sqrt{9 + 8}}{2}$.
The approximate values of x are 3.56 … and −0.56… .

The other two critical points are given by solving
$-(x^2 - 3x) = 2 \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow (x - 1)(x - 2) = 0$.
Therefore $x = 1$, $x = 2$.

The four critical points in ascending order are $\dfrac{3 - \sqrt{17}}{2}$, 1, 2, $\dfrac{3 + \sqrt{17}}{2}$.

By considering the graph, the intervals satisfying $|x^2 - 3x| > 2$ are
$x < \dfrac{3 - \sqrt{17}}{2}$, $1 < x < 2$, $x > \dfrac{3 + \sqrt{17}}{2}$.

EXERCISE 2E

Solve the following inequalities:

1 $|x-5|>2$ **2** $|4-x|\leqslant 1$ **3** $|x+3|<7$

4 $|x-2|\leqslant x$ **5** $|3-x|>x+4$ **6** $|2x+3|\leqslant x-2$

7 $|2x-5|\geqslant 2-x$ **8** $|2-5x|<x$ **9** $|x^2+4|\geqslant 5x$

10 $|2x^2-x|>1$ **11** $|7-3x^2|\leqslant 2x+1$

12 $|x^2+2x|\geqslant 7x+6$

Worked examination question

The function f is defined for all real values of x by

$$f(x)=|x|+|x-3|.$$

(a) For values of x such that $x<0$, show that $f(x)=3-2x$.

(b) Write down expressions for $f(x)$ in a form not involving modulus signs for each of the intervals:

(i) $x>3$; **(ii)** $0\leqslant x\leqslant 3$.

(c) Sketch the graph of f and write down the equation of its line of symmetry.

(d) State the range of f.

(e) Solve the equation $f(x)=4$.

(f) Explain whether it is possible to find an inverse of the function f. [A]

Solution

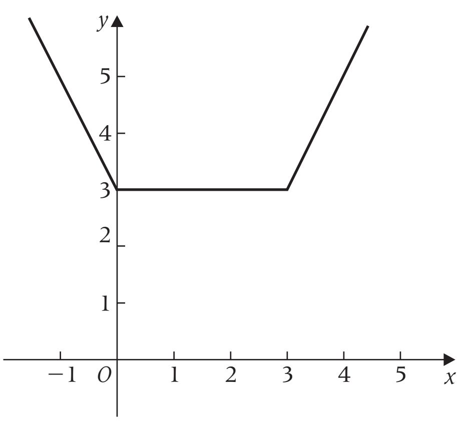
(a) When $x<0$, $|x|=-x$ and $|x-3|=-(x-3)=3-x$.
Hence $f(x)=-x+3-x=3-2x$.

(b) **(i)** When $x>3$, $|x|=x$ and $|x-3|=x-3$.
Hence $f(x)=x+x-3=2x-3$.

(ii) When $0\leqslant x\leqslant 3$, $|x|=x$ and $|x-3|=-(x-3)=3-x$.
Hence $f(x)=x+3-x=3$.

(c) The graph consists of the three sections defined above and is sketched opposite.

The line of symmetry has equation $x=\dfrac{3}{2}$.



(d) Since the function has least value equal to 3, the range is given by $f(x)\geqslant 3$.

(e) From the graph, the equation $f(x)=4$ has two solutions.
One of these is when $x<0\Rightarrow 3-2x=4\Rightarrow x=-\frac{1}{2}$.
The other is when $x>3\Rightarrow 2x-3=4\Rightarrow x=3\frac{1}{2}$.

(f) Since the graph of f is many-one, it does NOT have an inverse.

MIXED EXERCISE

1 The function f is defined for all real values of x by

$$f(x) = |2x - 3| - 1.$$

(a) Sketch the graph of $y = f(x)$. Indicate the coordinates of the points where the graph crosses the coordinate axes.

(b) State the range of $y = f(x)$.

(c) Find the values of x for which $f(x) = x$. [A]

2 Sketch the graphs of $y = |2x + 3|$ and $y = |2x - 5|$ on the same axes. Hence solve the inequality $|2x + 3| \geqslant |2x - 5|$.

3 The function f is defined by $f(x) = |x - 3|, x \in \mathbb{R}$.

(a) Sketch the graph of $y = f(x)$.

(b) Solve the inequality $|x - 3| < \frac{1}{2}x$. [A]

4 The functions f and g are defined for all real values of x by

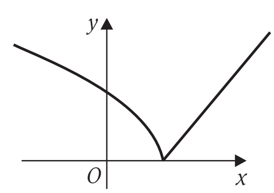
$$f(x) = x^2 - 10 \text{ and } g(x) = |x - 2|:$$

(a) Show that $ff(x) = x^4 - 20x^2 + 90$.
Find all the values of x for which $ff(x) = 26$.

(b) Show that $gf(x) = |x^2 - 12|$. Sketch the graph of $y = gf(x)$. Hence or otherwise, solve the equation $gf(x) = x$. [A]

5 **(a)** Determine the two values of x for which $|2x - 3| = |5 - x|$.

(b) The function f is defined for all real values of x. The graph of $y = |f(x)|$ is sketched opposite. Sketch two possible graphs of $y = f(x)$ on separate axes. [A]



6 **(a)** Sketch the graphs of
(i) $y = x^2 - 6x + 5$;
(ii) $y = |x^2 - 6x + 5|$.

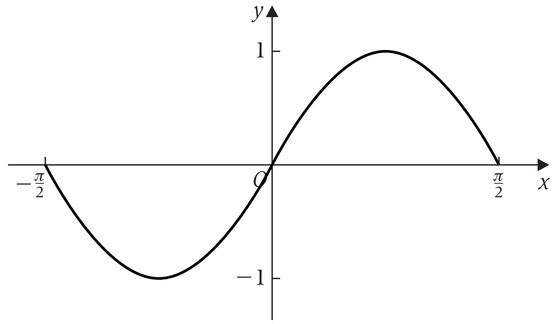
(b) Calculate the four roots of the equation $|x^2 - 6x + 5| = 3$, expressing the irrational solutions in surd form.

(c) Using this result and the sketch to **(a) (ii)**, or otherwise, solve the inequality

$$|x^2 - 6x + 5| \leqslant 3.$$ [A]

7 The diagram shows a sketch of the curve with equation

$$y = \sin 2x \text{ for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}.$$



(a) Draw on the same diagram sketches of the graphs with equations $y = |x|$ and $y = |\sin 2x|$ for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$.

(b) Hence state the number of times the graph of the curve with equation $y = |\sin 2x| - |x|$ intersects the x-axis in the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$. [A]

8 **(a)** Sketch the graph of $y = |2x - 4|$. Indicate the coordinates of the points where the graph meets the coordinate axes.

(b) **(i)** The line $y = x$ intersects the graph of $y = |2x - 4|$ at two points P and Q. Find the x-coordinates of the points P and Q.

(ii) Hence solve the inequality $|2x - 4| > x$.

(c) The graph of $y = |2x - 4| + k$ touches the line $y = x$ at only one point. Find the value of the constant k. [A]

9 A function f is defined for all real values of x by

$$f(x) = 3 - |2x - 1|.$$

(a) **(i)** Sketch the graph of $y = f(x)$. Indicate the coordinates of the points where the graph crosses the coordinate axes.

(ii) Hence show that the equation $f(x) = 4$ has no real roots.

(b) State the range of f.

(c) By finding the values of x for which $f(x) = x$, solve the inequality $f(x) < x$. [A]

Key point summary

1 A translation of $\begin{bmatrix} a \\ b \end{bmatrix}$ transforms the graph of $y = f(x)$ into the graph of $y = f(x - a) + b$. *p23*

2 The graph of $y = f(x)$ is transformed into the graph of $y = -f(x)$ by a reflection in the line $y = 0$ (the x-axis). *p23*

3 The graph of $y = f(x)$ is transformed into the graph of $y = f(-x)$ by a reflection in the line $x = 0$ (the y-axis). *p23*

4 The graph of $y = f(x)$ is transformed into the graph of $y = d\,f(x)$ by a stretch of scale factor d in the y-direction. *p24*

5 The graph of $y = f(x)$ is transformed into the graph of $y = f\left(\frac{x}{c}\right)$ by a stretch of scale factor c in the x-direction. *p24*

6 The **modulus function** $|x|$ is defined by

$$|x| \begin{cases} x \text{ when } x \geqslant 0. \\ -x \text{ when } x \leqslant 0. \end{cases}$$ *p30*

Test yourself

Test yourself	What to review
1 Describe geometrically how the curve: **(a)** $y = x^5$ is transformed into $y = (x + 1)^5 + 3$, **(b)** $y = \tan x$ is transformed into $y = \tan 4x$.	*Section 2.1*
2 Describe geometrically a sequence of transformations that transforms $y = x^2$ into $y = 3(x - 1)^2 + 4$.	*Section 2.2*
3 Describe geometrically a sequence of transformations that transforms $y = \sin x$ into $y = 3 + 2 \sin 4x$.	*Section 2.2*
4 The function f is defined for all values of x by $f(x) = \|x - 5\|$. **(a)** Express $f(x)$ in a form not involving modulus signs when $x < 5$. **(b)** Sketch the graph of $y = f(x)$, indicating any values where the graph meets the axes.	*Section 2.3*
5 **(a)** Sketch the graph of $y = 4 - x^2$. **(b)** Hence sketch the graph of $y = \|4 - x^2\|$. **(c)** State the number of roots of the equation $\|4 - x^2\| = 1$.	*Section 2.3*
6 Solve the equation $\|3x - 5\| = 2 - x$.	*Section 2.4*
7 Solve the inequality $\|2x - 7\| < 3 + x$.	*Section 2.5*

Test yourself ANSWERS

1 **(a)** Translation through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$;

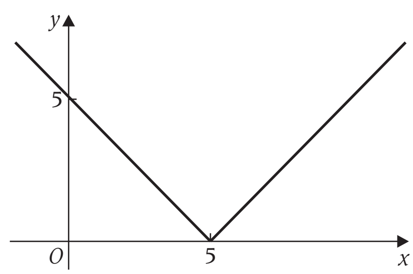
(b) Stretch in x-direction with scale factor $\frac{1}{4}$.

2 Translation of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, followed by stretch in y-direction with scale factor 3, followed by translation of $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

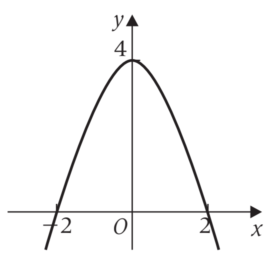
3 Stretch in x-direction with scale factor $\frac{1}{4}$ and stretch in y-direction with scale factor 2, followed by translation of $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

4 **(a)** $f(x) = 5 - x$, when $x < 5$;

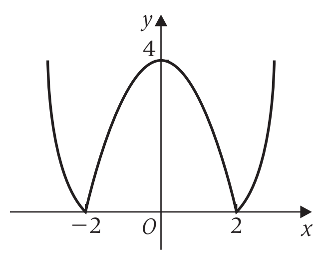
(b)



5 **(a)**



(b)



(c) 4 roots.

6 $x = 1\frac{1}{2}, 1\frac{3}{4}$.

7 $1\frac{1}{3} < x < 10$.