After completing this chapter you should be able to:

- Manipulate inequalities
- Determine the critical values of an inequality
- Find solutions of algebraic inequalities

1 Inequalities

Most applications of mathematics require the solution of inequalities at some stage

In manufacturing or business you will want to know what level of price will ensure that your profit is greater than your production costs – this means solving inequalities

1.1 You can manipulate inequalities to solve them

In C1 you learnt how to solve simple quadratic inequalities by rearranging them. The inequality sign can be treated like an equals sign as long as you do not divide or multiply both sides of the expression by a negative number.

There are three steps to solving inequalities.

Example 1

Solve $2x^2 < x + 3$

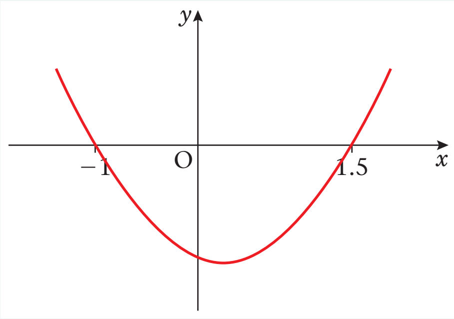
$2x^2 - x - 3 < 0$

$2x^2 - x - 3 = 0$

So $(2x - 3)(x + 1) = 0$

So the critical values are $x = \frac{3}{2}$ or -1

A sketch of $y = 2x^2 - x - 3$ gives



So the solution to $y = 2x^2 - x - 3 < 0$

is when $-1 < x < 1.5$

Step 1 is to find the critical values. Rearrange the expression and then replace the inequality symbol with an equals sign and solve.

Step 2 is to draw a sketch, or use a table of values to determine which sets of values satisfy the inequality.

Step 3 is to write down the answer by using the graph to interpret the inequality.

In FP2 you will dealing with algebraic fractions, and care must be taken when rearranging the inequality to make sure that you are not multiplying by a quantity that could be negative.

Example 2

Solve the inequality $\frac{x^2}{x - 2} < x + 1, x \neq 2$

Multiply both sides by $(x - 2)^2$

$(x - 2)^2 \times \frac{x^2}{x - 2}, (x - 2)^2 \times (x + 1)$

A natural first step would be to multiply both sides by $(x - 2)$ but we cannot be sure that this is positive. A simple solution is to multiply both sides of the inequality by $(x - 2)^2$ as this will always be positive.

$(x-2)^{\cancel{2}} \times \frac{x^2}{\cancel{(x-2)}} < (x-2)^2 \times (x+1)$

Do **not** aim to multiply out but cancel, collect terms on one side and **factorise**.

$(x-2)x^2 - (x+1)(x-2)^2 < 0$

$(x-2)(x^2 - (x+1)(x-2) < 0$

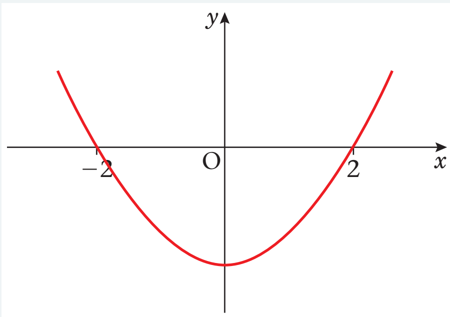
$(x-2)(x^2 - x^2 + x + 2) < 0$

or $(x-2)(x+2) < 0$

Critical values $x = \pm 2$

The sketch of $y = (x-2)(x+2)$ is

Now the problem is similar to those seen in C1. You find the critical values, draw a sketch and write down the answers.



The solution to $y = (x-2)(x+2) < 0$ is $-2 < x < 2$

The same approach can be used in more complicated situations.

Example 3

Solve the inequality $\frac{x}{x+1} \leqslant \frac{2}{x+3}$ $\quad x \neq -1, x \neq -3$

This time multiply both sides by

$(x+1)^2(x+3)^2$

In order to remove the fractions and guarantee that you are not multiplying by a negative quantity, use $(x+1)^2(x+3)^2$

So

$(x+1)^{\cancel{2}}(x+3)^2 \times \frac{x}{\cancel{(x+1)}} \leqslant \frac{2}{\cancel{(x+3)}} \times (x+1)^2(x+3)^{\cancel{2}}$

Cancel terms on each side.

$x(x+1)(x+3)^2 - 2(x+1)^2(x+3) \leqslant 0$

Collect terms on LHS

$(x+1)(x+3)(x(x+3) - 2(x+1)) \leqslant 0$

$(x+1)(x+3)(x^2 + x - 2) \leqslant 0$

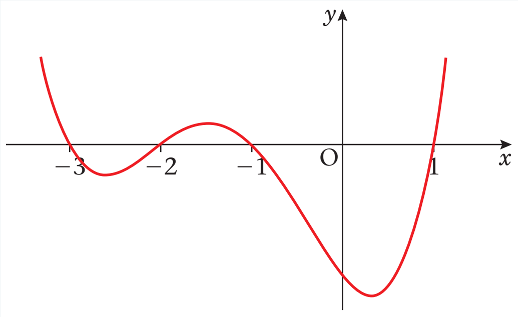
$(x+1)(x+3)(x+2)(x-1) \leqslant 0$

To find the critical values you need to solve. $(x+1)(x+3)(x+2)(x-1) = 0$

So the critical values are:

$x = -1, -3, -2$ or 1

A sketch of $y = (x + 1)(x + 3)(x + 2)(x - 1)$ is



So the solution to

$y = (x + 1)(x + 3)(x + 2)(x - 1) \leqslant 0$
is $-3 < x \leqslant -2$ or $-1 < x \leqslant 1$

The curve $y = (x + 1)(x + 3)(x + 2)(x - 1)$ is essentially an x^4 curve, so it starts in top left and ends in top right and passes through $x = -1, -3, -2$ or 1. The exact shape does not matter.

If the question has a $\leqslant$ as opposed to a $<$ then you must check carefully at the end whether the 'ends' of the set of values are valid. Since $x \neq -1$ or -3 these values must not be included, hence $<$ not $\leqslant$ is used.

Exercise 1A

Solve the following inequalities

1 $x^2 < 5x + 6$

2 $x(x + 1) \geqslant 6$

3 $\dfrac{2}{x^2 + 1} > 1$

4 $\dfrac{2}{x^2 - 1} > 1$

5 $\dfrac{x}{x - 1} \leqslant 2x \quad x \neq 1$

6 $\dfrac{3}{x + 1} < \dfrac{2}{x}$

7 $\dfrac{3}{(x + 1)(x - 1)} < 1$

8 $\dfrac{2}{x^2} \geqslant \dfrac{3}{(x - 1)(x - 2)}$

9 $\dfrac{2}{x - 4} < 3$

10 $\dfrac{3}{x + 2} > \dfrac{1}{x - 5}$

11 $\dfrac{3x^2 + 5}{x + 5} > 1$

12 $\dfrac{3x}{x - 2} > x$

13 $\dfrac{1 + x}{1 - x} > \dfrac{2 - x}{2 + x}$

14 $\dfrac{x^2 + 7x + 10}{x + 1} > 2x\ 1\ 7$

15 **a** $\dfrac{x + 1}{x^2} > 6$ **b** $\dfrac{x^2}{x + 1} > \dfrac{1}{6}$

1.2 You can use graphs to solve ineualities

Example 4

a On the same axes sketch the graphs of the curves with equations $y = \dfrac{7x}{3x + 1}$ and $y = 4 - x$

b Find the points of intersection of $y = \dfrac{7x}{3x + 1}$ and $y = 4 - x$.

c Solve $y = \dfrac{7x}{3x + 1} < 4 - x$

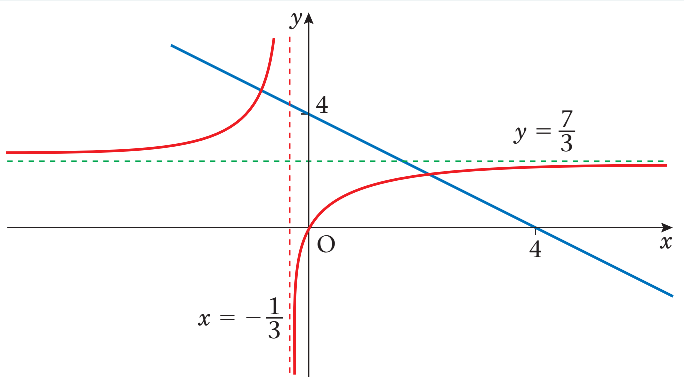
a $y = 4 - x$ is a straight line crossing the axes at (4, 0) and (0, 4).

$y = \frac{7x}{3x + 1}$ has a root at (0, 0)

There is a vertical asymptote at $x = -\frac{1}{3}$

There is a horizontal asymptote at $y = \frac{7}{3}$

So the sketch looks like this



For the sketches look for:

1 Intersections with the axes
2 Any vertical asymptotes
3 Any horizontal asymptotes – to find these $x \to \pm\infty$

Writing $y = \frac{7}{3}\left(\frac{3x}{3x+1}\right) = \frac{7}{3}\left(\frac{3x+1-1}{3x+1}\right)$ might help.

b $\frac{7x}{3x + 1} = 4 - x$

Multiply both sides by $3x + 1$

$7x = 12x + 4 - 3x^2 - x$

$3x^2 - 4x - 4 = 0$

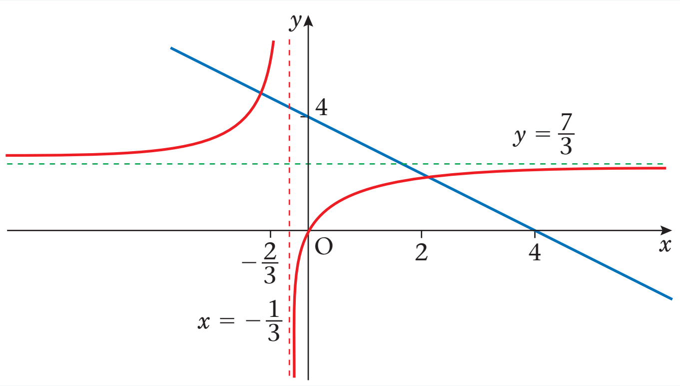
Multiply out and collect terms to form a quadratic equation.

$(3x + 2)(x - 2) = 0$

Solve the equation in the usual way – this one factorises

So $x = -\frac{2}{3}$ or 2

c Marking these points on the graph



Look on the sketch for the places where the line is above the curve.
These places will give the solution.

So the solution is

$x < -\frac{2}{3}$ or $-\frac{1}{3} < x < 2$

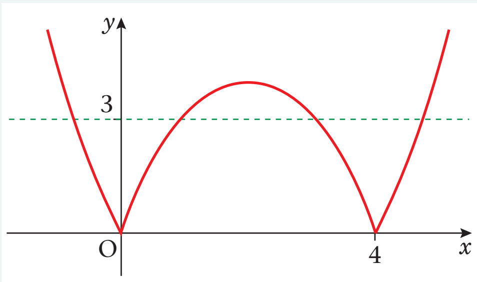
Notice how the vertical asymptote has an influence on the second part of the answer.

The sketching approach is particularly useful if the inequality involves the modulus function.

Example 5

Solve $|x^2 - 4x| < 3$

A sketch of $|y = x^2| - 4x$ and $y = 3$ looks like:



Since there is a modulus function sketch $y = |x^2 - 4x|$ and $y = 3$ on the same axes.

To find the critical values, solve $|x^2 - 4x| = 3$

To find the critical values, remember that there are two cases to consider when solving $|x^2 - 4x| = 3$.

$x^2 - 4x = 3 \Rightarrow x^2 - 4x - 3 = 0$

$$x = \frac{4 \pm \sqrt{16} + 12}{2}$$

Sometimes the quadratic formula may be required

$$x = \frac{4 \pm \sqrt{28}}{2}$$

$$x = \frac{4 \pm 2\sqrt{7}}{2}$$

$x = 2 \pm \sqrt{7}$

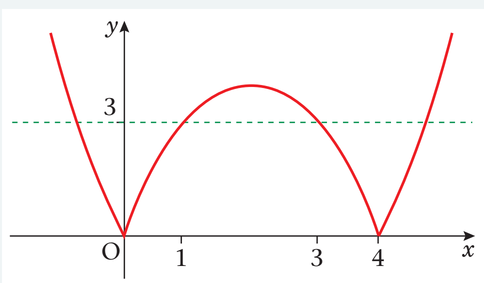
$2(x^2 - 4x) = 3 \Rightarrow x^2 - 4x + 3 = 0$

$(x - 3)(x - 1) = 0$

$x = 1$ or 3

Marking these values on the sketch:

You need to identify where the points of intersection are on the sketch.



So the solution is:

$2 - \sqrt{7} < x < 1$ or $3 < x < 2 + \sqrt{7}$

Finally write down the solution to the inequality – the points where the line $y = 3$ is above the curve.

Sometimes a little simple rearranging first can make the sketching much simpler

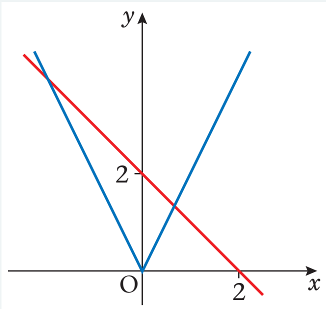
Example 6

Solve $|3x| + x \leqslant 2$

Rearranging gives:

$|3x| \leqslant 2 - x$

Sketching $y = 3x$ and $y = 2 - x$ gives



Critical values are given by:

$3x = 2 - x$

$4x = 2$

$x = \frac{1}{2}$

OR

$-3x = 2 - x$

$-2 = 2x$

$x = -1$

So the line is above $|3x|$ for

$-1 \leqslant x \leqslant \frac{1}{2}$

Sketching $y = |3x| + x$ is quite difficult so it is usually simpler to rearrange and isolate the modulus function.

Find the critical values in the usual way. Remember the two cases.

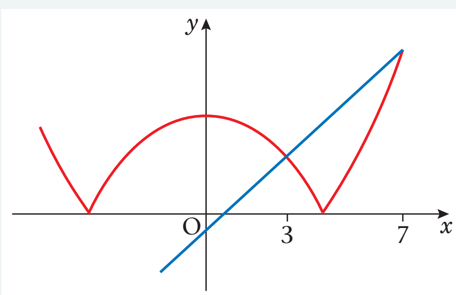
By considering the positions of the critical values, identify the places where the line is above the V shaped graph.

Sometimes care must be taken to identify the correct roots when solving the modulus equations.

Example 7

Solve the inequality $|x^2 - 19| \leqslant 5(x - 1)$

Sketching both graphs



First sketch the graphs

$x^2 - 19 = 5x - 5 \Rightarrow x^2 - 5x - 14 = 0$

$(x - 7)(x + 2) = 0$

$x = 7$ or -2

$-(x^2 - 19) = 5x - 5 \Rightarrow x^2 + 5x - 24 = 0$

$(x + 8)(x - 3) = 0$

$x = -8$ or 3

The line is above the curve when

$3 \leqslant x \leqslant 7$

Then find the critical values.

Solving the equations gives four values but the graphs only have two crossing points.
So the valid critical values are $x = 3$ and $x = 7$.

Finally write down the solution

Exercise 1B

Solve the following inequalities:

1 $|x - 6| > 6x$

2 $|t - 3| > t^2$

3 $|(x - 2)(x + 6)| < 9$

4 $|2x + 1| \geqslant 3$

5 $|2x| + x > 3$

6 $\dfrac{x + 3}{|x| + 1} < 2$

7 $\dfrac{3 - x}{|x| + 1} > 2$

8 $\left|\dfrac{x}{x + 2}\right| < 1 - x$

9 **a** On the same axes sketch the graphs of $y = \dfrac{1}{x}$ and $y = \dfrac{x}{x + 2}$.

b Solve $\dfrac{1}{x} > \dfrac{x}{x + 2}$

10 **a** On the same axes sketch the graphs of $y = \dfrac{1}{x - a}$ and $y = 4|x - a|$

b Solve, giving your answers in terms of the constant a, $\dfrac{1}{x - a}$, $4|x - a|$.

Mixed exercise 1C

1 Solve the inequality $|x^2 - 7| < 3(x + 1)$

2 Solve the inequality $\dfrac{x^2}{|x| + 6} < 1$

3 Find the set of value for which $|x - 1| > 6x - 1$ *E*

4 Find the complete set of values of x for which $|x^2 - 2| > 2x$ *E*

5 Find the set of values of x for which $\dfrac{x + 1}{2x - 3} < \dfrac{1}{x - 3}$ *E*

6 Solve $\dfrac{(x + 3)(x + 9)}{x - 1} > 3x - 5$ *E*

7 **a** Sketch, on the same axes, the graph with equation $y = |2x - 3|$, and the line with equation $y = 5x - 1$

b Solve the inequality $|2x - 3| < 5x - 1$

E

8 **a** Use algebra to find the exact solution of $|2x^2 + x - 6| = 6 - 3x$

b On the same diagram, sketch the curve with equation $y = |2x^2 + x - 6|$ and the line with equation $y = 6 - 3x$

c Find the set of values of x for which $|2x^2 + x - 6| > 6 - 3x$

E

9 **a** On the same diagram, sketch the graphs of $y = |x^2 - 4|$ and $y = |2x - 1|$, showing the coordinates of the points where the graphs meet the x-axis.

b Solve $|x^2 - 4| = |2x - 1|$, giving your answers in surd form where appropriate.

c Hence, or otherwise, find the set of values of x for which $|x^2 - 4| > |2x - 1|$

E

Summary of key points

1 Remember the three steps:
- find the critical values
- use a sketch or table to identify the solutions
- write down the answers checking carefully for validity if using $\leqslant$.

2 Try rearranging the inequality to make the sketching easier.

3 Remember you should only multiply by a quantity which is always positive.

4 A sketch is usually the best approach if a modulus function is involved.