Mixed simultaneous equations

You may be asked to solve simultaneous equations that involve a quadratic equation and a linear equation. You should solve simultaneous equations like this using **substitution**.

2 Substitution checklist

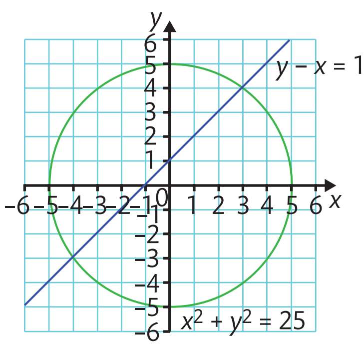
- Rearrange the linear equation to $x = \ldots$ or $y = \ldots$
- **Substitute** into the quadratic equation.
- Rewrite the resulting equation in the form $ax^2 + bx + c = 0$ and solve equation.
- Substitute each x-value into the linear equation to find the corresponding y-value.
- Write your final answer as coordinates of points.
- If the quadratic equation does not factorise you can either use the formula or complete the square.

You could substitute x or y into the quadratic equation. It is easier to substitute for y because there will be no fractions.

If the quadratic equation does not factorise you can either use the quadratic formula or complete the square. Have a look at pages 22 and 23.

5 Graphical method

You can solve simultaneous equations by drawing a graph. For example, to solve $x^2 + y^2 = 25$ and $y - x = 1$, rewrite the equations so that you can draw up tables of values. Then the graph looks like this.



The coordinates of the points of intersection give the solution to the simultaneous equations.

The solutions are $x = -4$, $y = -3$ and $x = 3$, $y = 4$, giving the points $(-4, -3)$ and $(3, 4)$.

10 Worked example Grade 8

1 Solve the simultaneous equations:

$2x - y = 7$

$x^2 + y^2 = 34$

You must show your working.

Do not use trial and improvement.

$y = 2x - 7$

$x^2 + (2x - 7)^2 = 34$

$x^2 + 4x^2 - 28x + 49 = 34$

$5x^2 - 28x + 15 = 0$

$(5x - 3)(x - 5) = 0$

$x = \frac{3}{5}$ and $x = 5$

$y = 2\left(\frac{3}{5}\right) - 7 = -\frac{29}{5}$ and $y = 2(5) - 7 = 3$

Solutions are $x = \frac{3}{5}$, $y = -\frac{29}{5}$ and $x = 5$, $y = 3$ or $\left(\frac{3}{5}, -\frac{29}{5}\right)$ and $(5, 3)$.

Always substitute from the linear equation into the quadratic equation.

2 Solve the simultaneous equations:

$y = 2x^2 + 3x - 2$

$y = 2x - 1$

You must show your working.

Do not use trial and improvement.

$2x^2 + 3x - 2 = 2x - 1$

$2x^2 + x - 1 = 0$

$(2x - 1)(x + 1) = 0$

$x = \frac{1}{2}$ and $x = -1$

So $y = 2\left(\frac{1}{2}\right) - 1 = 0$ and $y = 2(-1) - 1 = -3$

Solutions are $x = \frac{1}{2}$, $y = 0$ and $x = -1$, $y = -3$ or $\left(\frac{1}{2}, 0\right)$ and $(-1, -3)$.

As both equations have y as the subject, set the RHSs equal.

15 Exam-style practice Grade 8

Solve the following simultaneous equations.

(a) $y = 20 - 3x$
$y = 2x^2$ **[6 marks]**

(b) $x^2 + y^2 = 36$
$x = 2y + 6$ **[6 marks]**

(c) $x^2 + y^2 = 9$
$x + y = 2$ **[6 marks]**

Give your answers correct to 2 decimal places.

You must show your working.

Do not use trial and improvement.

Completing the square

Some quadratic equations cannot be factorised. To solve them, you **complete the square**.

2 Completing the square

This method for solving a quadratic equation involves rearranging it so that one side (the LHS) is a perfect square. If the number on the RHS is positive, you can take square roots on both sides to find two solutions.

You can use this formula to complete the square.

$x^2 \pm 2bx + c = (x \pm b)^2 - b^2 + c$

Look at the formula: $2b = 8$ so $b = 4$.

Look at the formula: $2b = 5$ so $b = \frac{5}{2}$.

5 Worked example — Grade 8

Solve $2x^2 - 12x + 3 = 0$

Leave your answer in the form $p \pm \sqrt{\frac{q}{r}}$.

Divide the quadratic by 2.

$x^2 - 6x + \frac{3}{2} = 0$

$(x - 3)^2 - 3^2 + \frac{3}{2} = 0$

$(x - 3)^2 - 9 + \frac{3}{2} = 0$

$(x - 3)^2 - \frac{15}{2} = 0$

$(x - 3)^2 = \frac{15}{2}$

$x - 3 = \pm\sqrt{\frac{15}{2}}$

$x = 3 \pm \sqrt{\frac{15}{2}}$

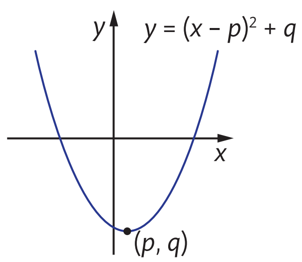
When you take the square root of both sides you need to add a ± sign in front of the square root on the RHS.

Completing the square is a very useful method for solving equations when you want an exact answer in surd form.

2 Minimum values of graphs

Quadratic graphs are curves. Completing the square can help you to work out the minimum point of the curve.

The minimum point of the curve with equation $y = (x - p)^2 + q$ is at (p, q).



5 Worked example — Grade 8

1 The expression $x^2 - 8x + 6$ can be written in the form $(x - p)^2 + q$ for all values of x. Work out the value of p and the value of q.

$x^2 - 8x + 6 = (x - 4)^2 - 4^2 + 6$

$= (x - 4)^2 - 16 + 6$

$= (x - 4)^2 - 10$

Hence, $p = 4$ and $q = -10$

2 The expression $x^2 + 5x - 3$ can be written in the form $(x + p)^2 + q$ for all values of x. Work out the value of p and the value of q.

$x^2 + 5x - 3 = (x + \frac{5}{2})^2 - (\frac{5}{2})^2 - 3$

$= (x + \frac{5}{2})^2 - \frac{25}{4} - 3$

$= (x + \frac{5}{2})^2 - \frac{25}{4} - \frac{12}{4}$

$= (x + \frac{5}{2})^2 - \frac{37}{4}$

Hence, $p = \frac{5}{2}$ and $q = -\frac{37}{4}$

Problem solving

When you are completing the square on quadratic equations you must first rearrange the equation into the form $ax^2 + bx + c$, then divide by a.

$(x - 3)^2 = x^2 - 6x + 9$, so

$x^2 - 6x = (x - 3)^2 - 9$

1 Completing the square checklist

- Make sure the coefficient of x^2 is 1.
- Remember the formula: $x^2 \pm 2bx + c = (x \pm b)^2 - b^2 + c$

10 Exam-style practice — Grade 8

1 The expression $x^2 - 8x + 5$ can be written in the form $(x - a)^2 + b$ for all values of x. Work out the values of a and b. **[3 marks]**

2 Solve $3x^2 - 18x + 5 = 0$.

Leave your answer in the form $p \pm \sqrt{\frac{q}{r}}$. **[3 marks]**

3 $x^2 + 2ax + 5 = (x + 3)^2 + b$

Work out the value of a and the value of b. **[3 marks]**

The quadratic formula

You can use the quadratic formula to solve quadratic equations.

5 The quadratic formula xy^2

When you cannot factorise a quadratic equation or complete the square, you can use the **quadratic formula**. The solutions of the quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To solve the equation, substitute the values of a, b and c into the formula.

You need to remember this formula for your exams.

2 Quadratic formula checklist

- Write out the quadratic equation in the form $ax^2 + bx + c = 0$.
- Write down the values of a, b and c.
- Show your substitution into the formula clearly.
- Use brackets when using negative numbers.
- The symbol ± means that you need to do two calculations.

5 Worked example Grade 7

Solve $3x^2 + 7x - 13 = 0$.

Give your answers correct to 3 significant figures.

$a = 3 \quad b = 7 \quad c = -13$

$$x = \frac{-7 \pm \sqrt{7^2 - (4 \times 3 \times -13)}}{2 \times 3}$$

$$x = \frac{-7 \pm \sqrt{205}}{6}$$

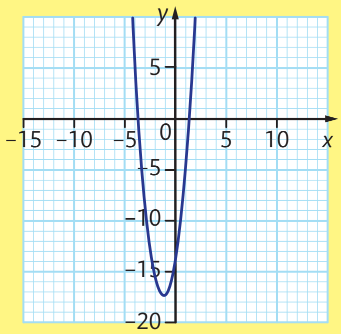
$$x = \frac{-7 - \sqrt{205}}{6} \quad \text{or} \quad x = \frac{-7 + \sqrt{205}}{6}$$

$x = -3.55$ or $x = 1.22$

The quadratic equation does not factorise, so you need to use the quadratic formula.

If the question asks you to find solutions to a given degree of accuracy it is a clue that you should use the quadratic formula.

The graph of $y = 3x^2 + 7x - 13$ cuts the x-axis at $x = -3.55$ and $x = 1.22$.



2 The discriminant

In the quadratic formula the expression under the square root sign is called the **discriminant**. The discriminant tells you how many real solutions a quadratic equation has.

$b^2 - 4ac > 0$: equation has two different solutions

$b^2 - 4ac = 0$: equation has one solution

$b^2 - 4ac < 0$: equation has no solutions

2 Worked example Grade 7

Show that the quadratic equation $4x^2 - 3x + 9 = 0$ has no real solutions.

$a = 4 \quad b = -3 \quad c = 9$

Using the discriminant:

$b^2 - 4ac = (-3)^2 - (4 \times 4 \times 9)$

$= 9 - 144$

$= -135 < 0$

There are no real solutions.

15 Exam-style practice Grade 7

1. Solve $\frac{1}{x} - 3x = 4$.

   Give your answers correct to 2 decimal places. **[4 marks]**

2. Work out the number of solutions of each quadratic equation.

   (a) $6x^2 + 2x - 3 = 0$ **[2 marks]**

   (b) $2x^2 - 8x + 11 = 0$ **[2 marks]**

3. The quadratic formula gives the following information

   $$x = \frac{13 \pm \sqrt{169 - 80}}{8}$$

   Work out the quadratic equation solved. Give your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers. **[3 marks]**

Linear inequalities

Inequalities are used to compare values. They show when one value is greater than or less than another value, using the symbols $<$ (is less than), $>$ (is greater than), $\leqslant$ (is less than or equal to) and $\geqslant$ (is greater than or equal to).

5 Solving inequalities

Inequalities behave in a similar way to equations. The same rules are used to solve inequalities as for equations. Whatever you do on one side, you must do on the other.

$2x - 5 < 7 \qquad 2x - 5 + 5 < 7 + 5$

$2x < 12 \qquad 2x \div 2 < 12 \div 2$

$x < 6$

The solution set is $\{x \mid x$ is a real number and $x < 6\}$.

The aim is to get the unknown on its own, on one side of the inequality, and a number, or an algebraic expression that does not include this unknown, on the other side.

2 Worked example — Grade 5

Given that $-3 < n \leqslant 3$ and n is an integer, write down all the possible values of n.

Values of n are $-2, -1, 0, 1, 2, 3$.

The solution set is $\{n \mid n$ is an integer and $-2, -1, 0, 1, 2\}$.

This is really two inequalities: $n > -3$ and $n \leqslant 3$

Remember that an open circle means $<$ or $>$ and a solid circle means $\leqslant$ or $\geqslant$.

5 Worked example — Grade 5

1 Solve $3(x - 2) > 15$.

$3x - 6 > 15$

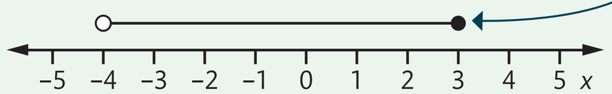
$3x > 21$

$x > 7$

Expand the bracket and then simplify the inequality.

The solution set is $\{x \mid x$ is a real number and $x > 7\}$.

2



Write down the inequality shown in the diagram.

$-4 < x \leqslant 3$

The solution set is $\{x \mid x$ is a real number and $-4 < x \leq 3\}$.

2 The golden rules

If you multiply or divide by a positive number then the inequality sign does not change.

If you multiply or divide by a negative number then you have to reverse the inequality.

$-x < 10$ multiplied by -1 = $x > -10$

2 Inequalities checklist

- ✓ Solve an inequality in exactly the same way as an equation.
- ✓ Multiplying or dividing through by a negative number reverses the sign.
- ✓ $x < 5$ means 'x is less than 5', so 5 is not included in possible values; $x \leqslant 5$ means 5 is included.

5 Worked example — Grade 6

Solve $\dfrac{2 - x}{4} < 3x - 4$.

$2 - x < 4(3x - 4)$

$2 - x < 12x - 16$

Expand the bracket and then solve the inequality.

$-x - 12x < -16 - 2$

$-13x < -18$

$x > \dfrac{-18}{-13}$

Reverse the inequality when dividing by a negative number.

$x > \dfrac{18}{13}$

The solution set is $\{x \mid x$ is a real number and $x > \dfrac{18}{13}\}$.

10 Exam-style practice — Grades 5–7

1 Given that x and y are integers such that

$4 < x < 8 \qquad 5 < y < 10 \qquad x + y = 14$

list all the possible values of x. **[2 marks]**

2 What are the integer values of x that satisfy both of these inequalities:

$3x - 12 > 4$ and $2x - 3 \leqslant 14$? **[3 marks]**

3 **(a)** Solve the inequality $\dfrac{2x - 3}{3} > \dfrac{5 - x}{2}$. **[3 marks]**

(b) x is an integer. Write down the smallest value of x that satisfies $\dfrac{2x - 3}{3} > \dfrac{5 - x}{2}$. **[1 mark]**

Quadratic inequalities

Make sure you are confident solving quadratic equations before you revise quadratic inequalities. For a reminder about quadratic equations, have a look at page 20.

5 Solving inequalities

The secret of solving a quadratic inequality is to solve the corresponding quadratic equation and then sketch the graph. For example, to solve:

$x^2 + 5x - 24 \geqslant 0$

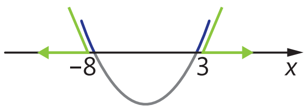
factorise the left-hand side:

$(x + 8)(x - 3) \geqslant 0$

The **critical values** are $x = -8$ and $x = 3$.

These values occur where the curve with equation $y = (x + 8)(x - 3)$ crosses the x-axis.

Always draw a sketch of your curve, using the critical values to show where the curve cuts the x-axis.



Highlight the area where $y \geqslant 0$.

$x \leqslant -8$ and $x \geqslant 3$

The solution set is $\{x \mid x$ is a real number and $-8 \leqslant x \geqslant 3\}$.

There are two regions so write the answer as two inequalities.

5 Worked example — Grade 8

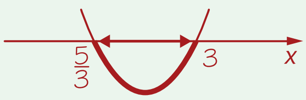
Solve the inequality $3(x^2 + 5) < 14x$.

$3x^2 + 15 < 14x$

$3x^2 - 14x + 15 < 0$

$(3x - 5)(x - 3) < 0$

Critical values are $x = \frac{5}{3}$ and $x = 3$.



$x > \frac{5}{3}$ and $x < 3$ so $\frac{5}{3} < x < 3$

The solution set is $\{x \mid x$ is a real number and $\frac{5}{3} \leqslant x \leqslant 3\}$.

First expand the bracket and then rearrange to the form $ax^2 + bx + c < 0$.

The required region is where the graph is **below** the x-axis.

5 Worked example — Grade 8

Solve the inequality $x^2 \leqslant 16$.

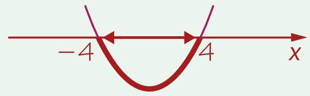
Represent your answer on a number line.

$x^2 \leqslant 16$

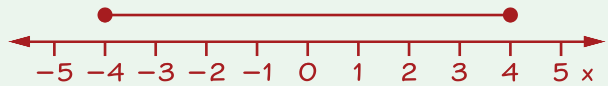
$x^2 - 16 \leqslant 0$

$(x - 4)(x + 4) \leqslant 0$

Critical values are $x = -4$ and $x = 4$.



$x \geqslant -4$ and $x \leqslant 4$ so $-4 \leqslant x \leqslant 4$



The solution set is $\{x \mid x$ is a real number and $-4 \leqslant x \leqslant 4\}$.

2 Methods

- Factorise and solve the quadratic to find the critical values.
- Sketch a graph.
- Highlight the area above or below the x-axis.
- Write out the **two** inequalities.

15 Exam-style practice — Grades 5–8

1 **(a)** Solve the inequality $x^2 - 9 > 0$. **[2 marks]**

(b) Represent your answer on a number line. **[1 mark]**

2 Solve these inequalities.

(a) $x^2 > 9x - 20$ **[3 marks]** **(b)** $x^2 > 3(x + 6)$ **[3 marks]** **(c)** $3x^2 < 2(x + 4)$ **[3 marks]**

3 **(a)** Sketch the graph of $y = x^2 - 2x - 3$. **[2 marks]**

(b) Hence, or otherwise, solve the inequality

$x^2 - 2x - 3 \geqslant 0$ **[2 marks]**

Arithmetic sequences

An arithmetic or linear sequence is a sequence of numbers in which the difference between consecutive terms is constant.

5 Finding the *n*th term

An example of an **arithmetic sequence** is

–4 1 6 11 16

+5 +5 +5 +5

The difference between one term and the next term is always +5.

Here is another arithmetic sequence:

8 5 2 –1 –4

–3 –3 –3 –3

You can work out a formula to calculate any term of an arithmetic sequence. This is called the *n*th term, where *n* is an integer, and is of the form $an \pm b$.

A quick method to find *b*

You can use this method to work out the formula for the *n*th term.

*n*th term = difference × *n* + zero term

where zero term = 1st term – difference

Using the zero term is a quick way to find *b*, the number that needs to be added or subtracted to the first part of the formula.

5 Worked example Grade 5

Here are the first four terms of an arithmetic sequence.

11 17 23 29

Work out, in terms of *n*, an expression for the *n*th term of this arithmetic sequence.

11 17 23 29

+6 +6 +6

Common difference = + 6

Zero term = 11 – 6 = 5

So *n*th term = 6*n* + 5

Check: 3rd term = 23

6 × 3 + 5 = 18 + 5 = 23 ✓

5 Checking a term

You may be asked to work out if a number is part of a given sequence. For example, here are the first five terms of an arithmetic sequence.

3 7 11 15 19

Is 93 a term in the sequence?

Start with the *n*th term.

The *n*th term of this sequence is $4n - 1$.

Set the *n*th term equal to 93 and solve the equation.

$4n - 1 = 93$

$4n = 94$

$n = 23.5$

If your answer is an **integer** (a whole number) then the term is in the sequence. Otherwise, it is not.

93 is not in the sequence.

Always write a conclusion.

5 Worked example Grade 5

Here are the first five terms of an arithmetic sequence.

5 12 19 26 33

(a) Write down an expression, in terms of *n*, for the *n*th term of the sequence.

5 12 19 26 33

+7 +7 +7 +7

Zero term = 5 – 7 = –2

*n*th term = 7*n* – 2

(b) Is 82 a term in the sequence?
You must give a reason for your answer.

$7n - 2 = 82$

$7n = 84$

$n = 12$

Work backwards to calculate the zero term of the sequence.

Yes 82 is in the sequence, because *n* is an integer.

10 Exam-style practice Grade 5

1 Here are the first five terms of an arithmetic sequence.

8 5 2 –1 –4

Circle the expression for the *n*th term of the sequence.

3*n* + 5 n – 3 11 – 3*n* 8 – 3*n* **[1 mark]**

2 Here are the first five terms of an arithmetic sequence.

3 8 13 18 23

(a) Explain why the number 162 cannot be a term in this sequence. **[1 mark]**

(b) Write down an expression, in terms of *n*, for the *n*th term of the sequence. **[2 marks]**