E/P **6** $f(x) = \dfrac{6 + 3x - x^2}{x^3 + 2x^2}$, $x > 0$

a Express $f(x)$ in partial fractions. **(4 marks)**

b Hence find the exact value of $\int_2^4 \dfrac{6 + 3x - x^2}{x^3 + 2x^2}\,dx$, writing your answer in the form $a + \ln b$, where a and b are rational numbers to be found. **(5 marks)**

E/P **7** $\dfrac{32x^2 + 4}{(4x + 1)(4x - 1)} \equiv A + \dfrac{B}{4x + 1} + \dfrac{C}{4x - 1}$

a Find the value of the constants A, B and C. **(4 marks)**

b Hence find the exact value of $\int_1^2 \dfrac{32x^2 + 4}{(4x + 1)(4x - 1)}\,dx$ writing your answer in the form $2 + k\ln m$, giving the values of the rational constants k and m. **(5 marks)**

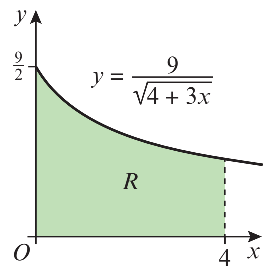
11.8 Finding areas

You need to be able to use the integration techniques from this chapter to find areas under curves.

Example 22

The diagram shows part of the curve $y = \dfrac{9}{\sqrt{4 + 3x}}$

The region R is bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$, as shown in the diagram. Use integration to find the area of R.



$$\text{Area} = \int_0^4 \frac{9}{\sqrt{4 + 3x}}\,dx$$

$$= 9\int_0^4 (4 + 3x)^{-\frac{1}{2}}\,dx$$

Remember $\dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$

$$= 6\left[(4 + 3x)^{\frac{1}{2}}\right]_0^4$$

Use the chain rule in reverse. If $y = (4 + 3x)^{\frac{1}{2}}$, $\dfrac{dy}{dx} = \dfrac{3}{2}(4 + 3x)^{-\frac{1}{2}}$. Adjust for the constant.

$$= 6\left((4 + 3 \times 4)^{\frac{1}{2}} - (4 + 3 \times 0)^{\frac{1}{2}}\right)$$

Substitute the limits.

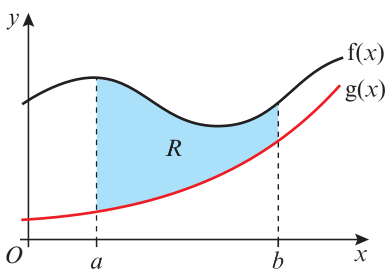
$$= 6(\sqrt{16} - \sqrt{4})$$

$$= 12$$

You don't need to give units when finding areas under graphs in pure maths.

- **The area bounded by two curves can be found using integration:**

$$\textbf{Area of } R = \int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

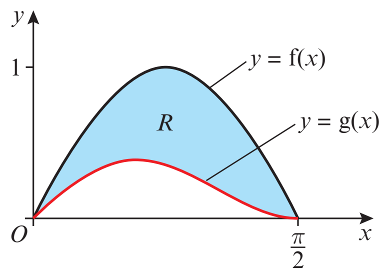


Watch out You can only use this formula if the two curves do not intersect between a and b.

Example 23

The diagram shows part of the curves $y = f(x)$ and $y = g(x)$, where $f(x) = \sin 2x$ and $g(x) = \sin x \cos^2 x$, $0 \leqslant x \leqslant \frac{\pi}{2}$

The region R is bounded by the two curves. Use integration to find the area of R.



$$\text{Area} = \int_a^b (f(x) - g(x))\,dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin 2x - \sin x \cos^2 x)\,dx$$

$$= \left[-\tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos^3 x\right]_0^{\frac{\pi}{2}}$$

$$= \left(-\tfrac{1}{2}(-1) + \tfrac{1}{3}(0)\right) - \left(-\tfrac{1}{2} + \tfrac{1}{3}\right) = \tfrac{2}{3}$$

The region R is bounded by two curves.

Substitute the limits and functions given in the question.

Online Explore the area between two curves using technology.

You can use integration to find the area under a curve defined by parametric equations. It is often easier to integrate with respect to the parameter.

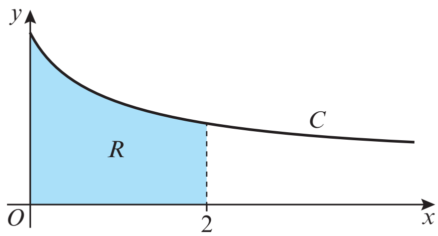
Links For a parametric curve, x and y are given as functions of a parameter, t. ← Chapter 8

Example 24

The curve C has parametric equations

$x = t(1 + t)$, $y = \dfrac{1}{1 + t}$, $t \geqslant 0$

Find the exact area of the region R, bounded by C, the x-axis and the lines $x = 0$ and $x = 2$.



$$\text{Area} = \int y\,dx = \int y\frac{dx}{dt}\,dt$$

$x = t(1 + t)$, so $\dfrac{dx}{dt} = 1 + 2t$

When $x = 0$, $t(1 + t) = 0$, so $\underline{t = 0}$ or $t = -1$

When $x = 2$, $t(1 + t) = 2$

$t^2 + t - 2 = 0$

$(t + 2)(t - 1) = 0$, so $t = -2$ or $\underline{t = 1}$

$$\text{So Area} = \int_0^1 y\frac{dx}{dt}\,dt = \int_0^1 \frac{1}{1 + t}(1 + 2t)\,dt$$

$$= \int_0^1 \left(2 - \frac{1}{1 + t}\right) dt$$

$$= [2t - \ln|1 + t|]_0^1$$

$$= (2 - \ln 2) - (0 - \ln 1)$$

$$= 2 - \ln 2$$

Use a change of variable to write the integral in terms of the parameter, t. Using the chain rule, you can replace dx with $\dfrac{dx}{dt}\,dt$.

$x = t + t^2$

Watch out You will be integrating **with respect to t** so you need to convert the limits from values of x to values of t. Use the parametric equation for x, and choose solutions that are within the domain of the parameter, $t \geqslant 0$.

Write $\dfrac{1 + 2t}{1 + t}$ in the form $A + \dfrac{B}{1 + t}$

Dividing $2t + 1$ by $t + 1$ gives 2 with remainder -1.

Exercise 11H

1 Find the area of the finite region R bounded by the curve with equation $y = \mathrm{f}(x)$, the x-axis and the lines $x = a$ and $x = b$.

a $\mathrm{f}(x) = \frac{2}{1+x}$; $a = 0$, $b = 1$ **b** $\mathrm{f}(x) = \sec x$; $a = 0$, $b = \frac{\pi}{3}$ **c** $\mathrm{f}(x) = \ln x$; $a = 1$, $b = 2$

d $\mathrm{f}(x) = \sec x \tan x$; $a = 0$, $b = \frac{\pi}{4}$ **e** $\mathrm{f}(x) = x\sqrt{4 - x^2}$; $a = 0$, $b = 2$

2 Find the exact area of the finite region bounded by the curve $y = \mathrm{f}(x)$, the x-axis and the lines $x = a$ and $x = b$ where:

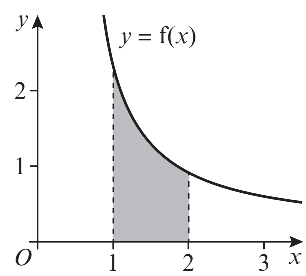
a $\mathrm{f}(x) = \frac{4x - 1}{(x+2)(2x+1)}$; $a = 0$, $b = 2$ **b** $\mathrm{f}(x) = \frac{x}{(x+1)^2}$; $a = 0$, $b = 2$

c $\mathrm{f}(x) = x \sin x$; $a = 0$, $b = \frac{\pi}{2}$ **d** $\mathrm{f}(x) = \cos x\sqrt{2\sin x + 1}$; $a = 0$, $b = \frac{\pi}{6}$

e $\mathrm{f}(x) = xe^{-x}$; $a = 0$, $b = \ln 2$

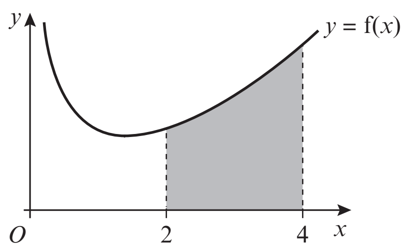
E 3 The diagram shows a sketch of the curve with equation, $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \frac{4x + 3}{(x+2)(2x-1)}$, $x > \frac{1}{2}$

Find the area of the shaded region bounded by the curve, the x-axis and the lines $x = 1$ and $x = 2$. **(7 marks)**



E 4 The diagram shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = e^{0.5x} + \frac{1}{x}$, $x > 0$.

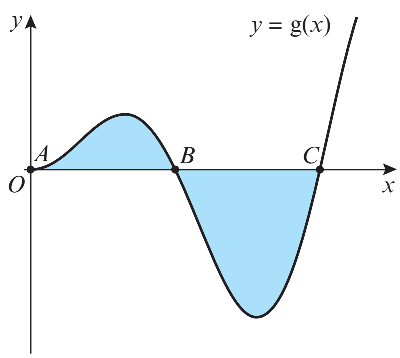
Find the area of the shaded region bounded by the curve, the x-axis and the lines $x = 2$ and $x = 4$. **(7 marks)**



P 5 The diagram shows a sketch of the curve with equation $y = \mathrm{g}(x)$, where $\mathrm{g}(x) = x \sin x$.

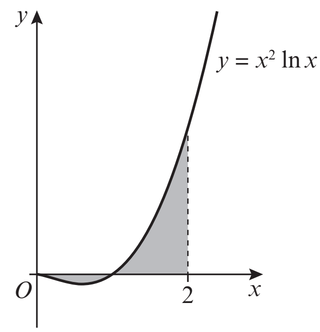
a Write down the coordinates of points A, B and C.

b Find the area of the shaded region.



Watch out Find the area of each region separately and then add the answers. Remember areas cannot be negative, so take the absolute value of any negative area.

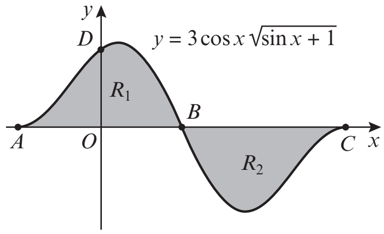
E/P **6** The diagram shows a sketch of the curve with equation $y = x^2 \ln x$. The shaded region is bounded by the curve, the x-axis and the line $x = 2$.



a Use integration by parts to find $\int x^2 \ln x \, dx$. **(3 marks)**

b Hence find the exact area of the shaded region, giving your answer in the form $\frac{2}{3}(a \ln 2 + b)$, where a and b are integers. **(5 marks)**

E/P **7** The diagram shows a sketch of the curve with equation $y = 3\cos x\sqrt{\sin x + 1}$.



a Find the coordinates of the points A, B, C and D. **(3 marks)**

b Use a suitable substitution to find

$\int 3\cos x\sqrt{\sin x + 1} \, dx$ **(5 marks)**

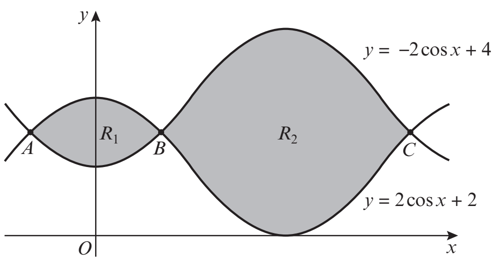
c Show that the regions R_1 and R_2 have the same area, and find the exact value of this area in the form $\sqrt{a}$, where a is a positive integer to be found. **(3 marks)**

P **8** $f(x) = x^2$ and $g(x) = 3x - x^2$

a On the same axes, sketch the graphs of $y = f(x)$ and $y = g(x)$, and find the coordinates of any points of intersection of the two curves.

b Find the area of the finite region bounded by the two curves.

E/P **9** The diagram shows a sketch of part of the curves with equations $y = 2\cos x + 2$ and $y = -2\cos x + 4$.

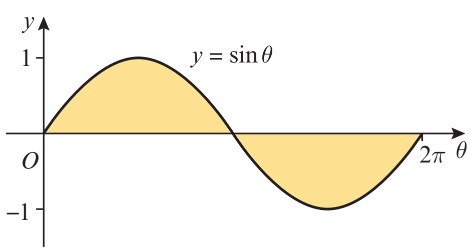


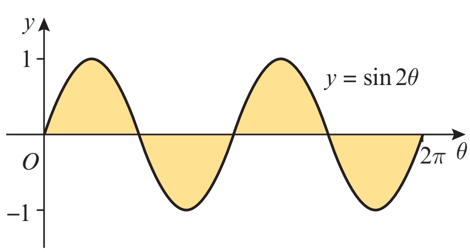
a Find the coordinates of the points A, B and C. **(2 marks)**

b Find the area of region R_1 in the form $a\sqrt{3} + \frac{b\pi}{c}$, where a, b and c are integers to be found. **(4 marks)**

c Show that the ratio of $R_2 : R_1$ can be expressed as $(3\sqrt{3} + 2\pi) : (3\sqrt{3} - \pi)$. **(5 marks)**

P **10** The diagrams show the curves $y = \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$ and $y = \sin 2\theta$, $0 \leqslant \theta \leqslant 2\pi$.
By choosing suitable limits, show that the total shaded area in the first diagram is equal to the total shaded area in the second diagram, and state the exact value of this shaded area.





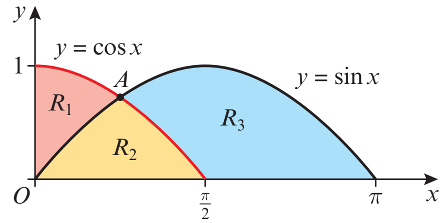
(P) **11** The diagram shows parts of the graphs of $y = \sin x$ and $y = \cos x$.

a Find the coordinates of point A.

b Find the areas of:

i R_1 **ii** R_2 **iii** R_3

c Show that the ratio of areas $R_1 : R_2$ can be written as $\sqrt{2} : 2$.

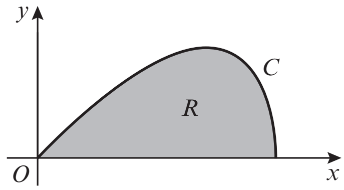


(P) **12** The curve C has parametric equations $x = t^3$, $y = t^2$, $t \geqslant 0$. Show that the exact area of the region bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$ is $k\sqrt[3]{2}$, where k is a rational constant to be found.

(E/P) **13** The curve C has parametric equations

$$x = \sin t,\ y = \sin 2t,\ 0 \leqslant t \leqslant \frac{\pi}{2}$$

The finite region R is bounded by the curve and the x-axis. Find the exact area of R. **(6 marks)**



(E/P) **14** This graph shows part of the curve C with parametric equations $x = (t + 1)^2$, $y = \frac{1}{2}t^3 + 3$, $t \geqslant -1$

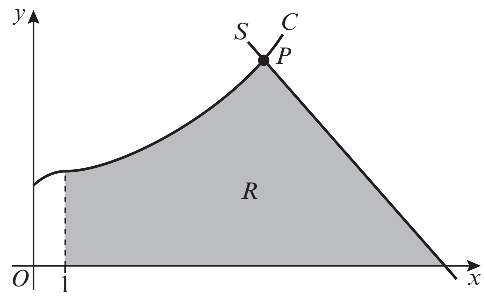
P is the point on the curve where $t = 2$.

The line S is the normal to C at P.

a Find an equation of S. **(5 marks)**

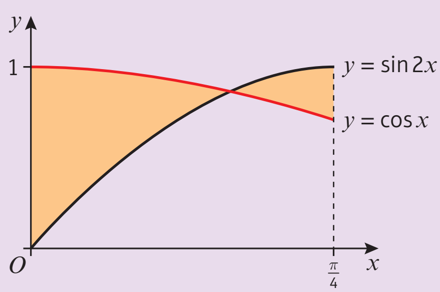
The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

b Using integration, find the area of R. **(5 marks)**



Challenge

The diagram shows the curves $y = \sin 2x$ and $y = \cos x$, $0 \leqslant x \leqslant \frac{\pi}{4}$



Find the exact value of the total shaded area on the diagram.