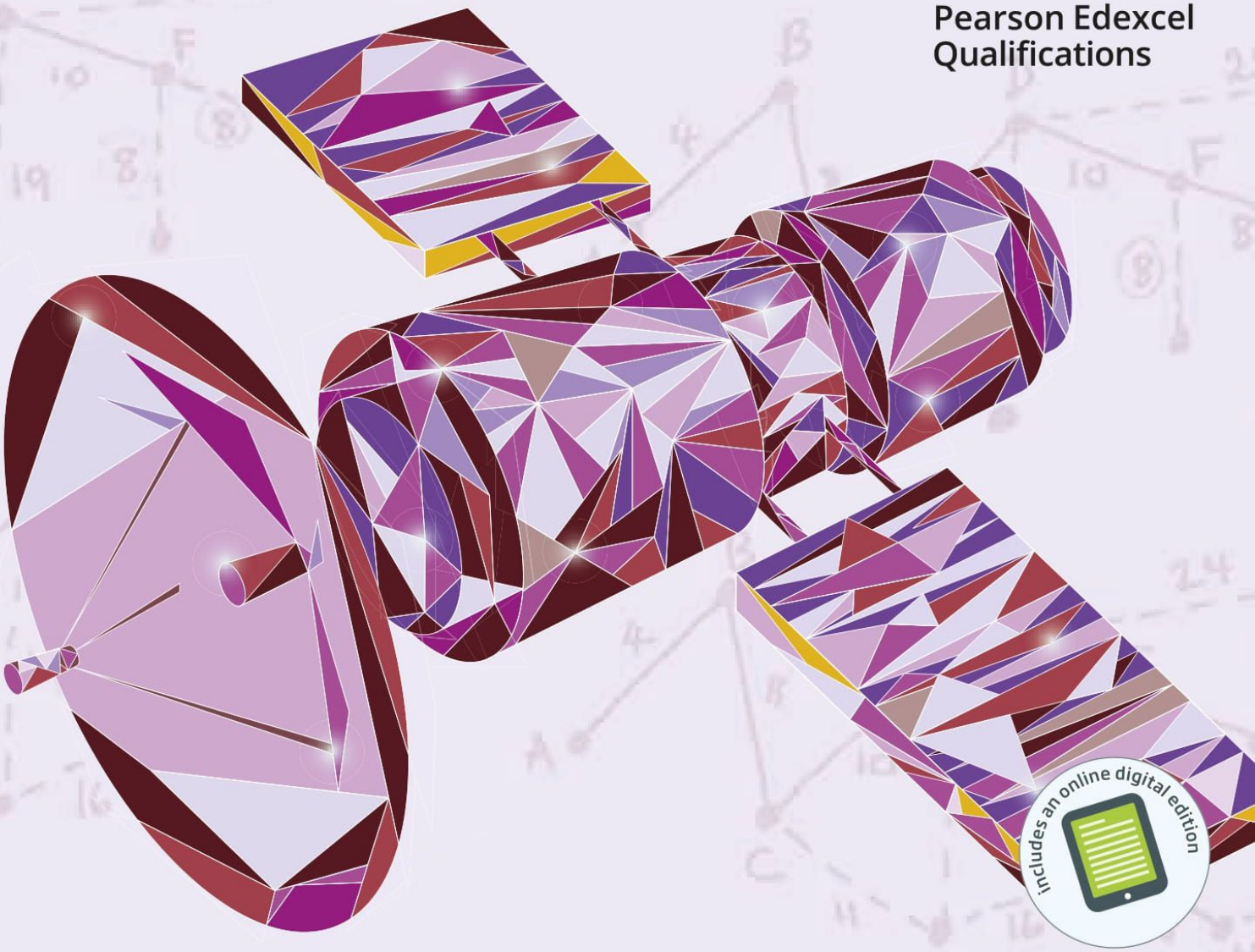
11 - 19 PROGRESSION

Endorsed for
Pearson Edexcel Qualifications



Pearson Edexcel AS and A level Further Mathematics

Decision Mathematics 2

D2

Pearson

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Overarching themes

The following three overarching themes have been fully integrated throughout the Pearson Edexcel AS and A level Mathematics series, so they can be applied alongside your learning and practice.

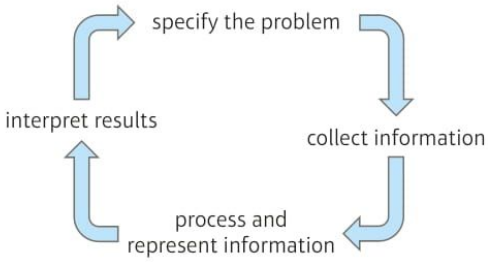
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols
- Dedicated sections on mathematical proof explain key principles and strategies
- Opportunities to critique arguments and justify methods

2. Mathematical problem solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Structured and unstructured questions to build confidence
- Challenge boxes provide extra stretch

The Mathematical Problem-solving cycle

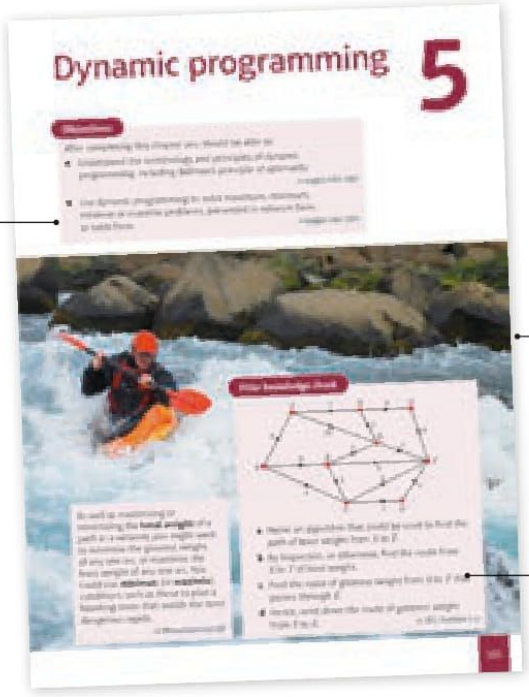


3. Mathematical modelling

- Dedicated modelling sections in relevant topics provide plenty of practice where you need it
- Examples and exercises include qualitative questions that allow you to interpret answers in the context of the model
- Dedicated chapter in Statistics & Mechanics Year 1/AS explains the principles of modelling in mechanics

Finding your way around the book

Access an online digital edition using the code at the front of the book.



Each chapter starts with a list of objectives

The real world applications of the maths you are about to learn are highlighted at the start of the chapter with links to relevant questions in the chapter

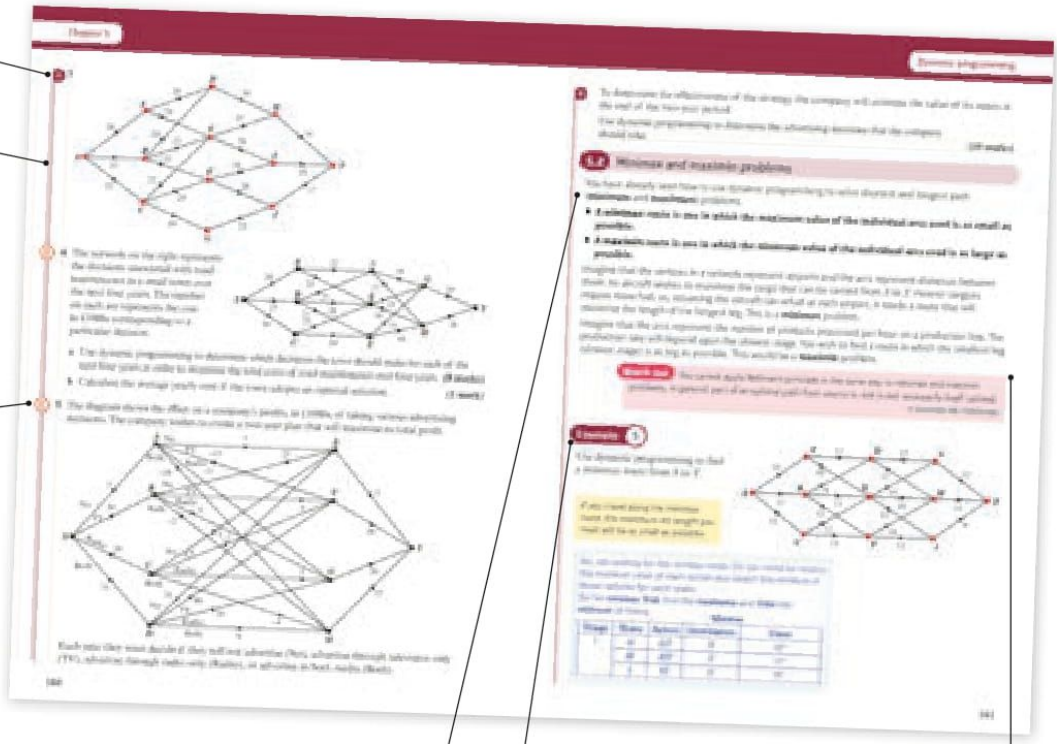
The *Prior knowledge check* helps make sure you are ready to start the chapter

A level content is clearly flagged

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams

Challenge boxes give you a chance to tackle some more difficult questions



Exam-style questions are flagged with (E)

Problem-solving questions are flagged with (P)

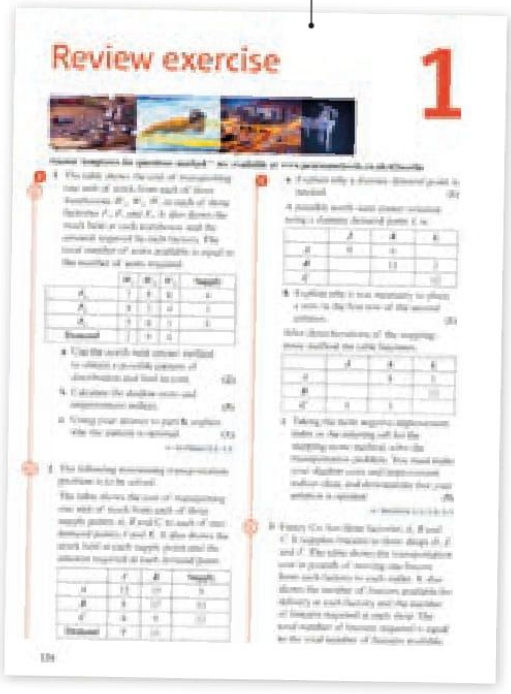
Each chapter ends with a *Mixed exercise* and a *Summary of key points*

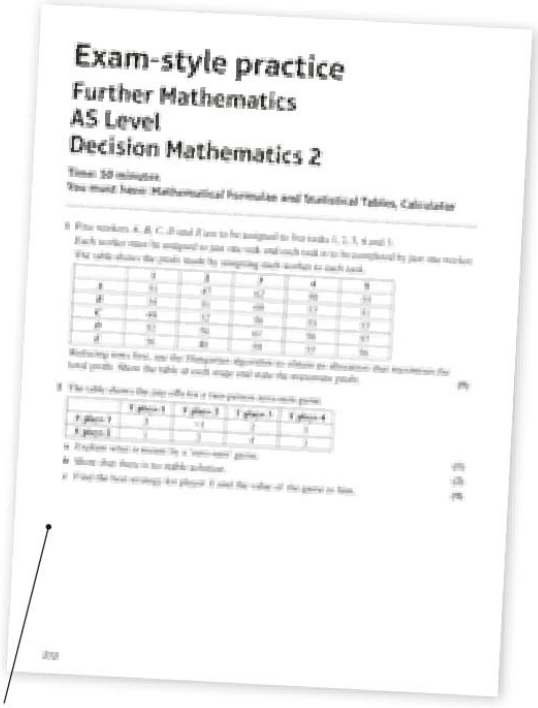
Each section begins with explanation and key learning points

Step-by-step worked examples focus on the key types of questions you'll need to tackle

Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

Every few chapters a *Review exercise* helps you consolidate your learning with lots of exam-style questions





AS and A level practice papers at the back of the book help you prepare for the real thing.

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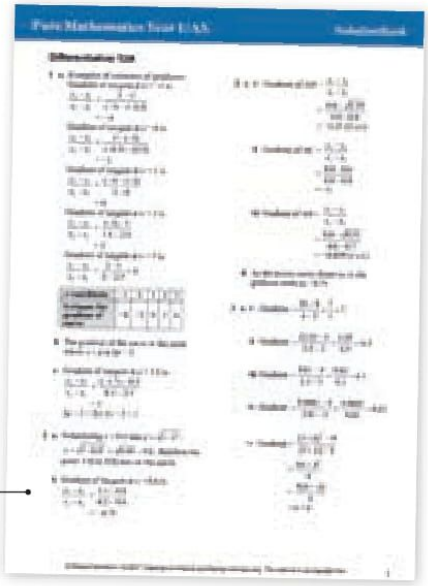
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Use of technology

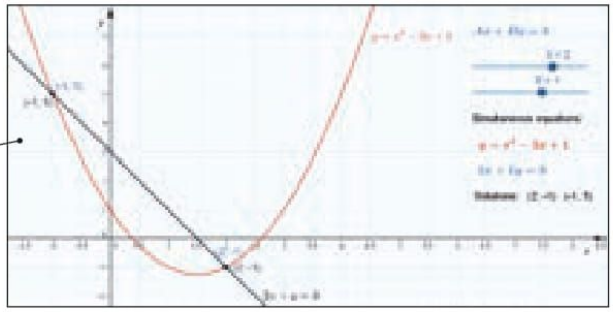
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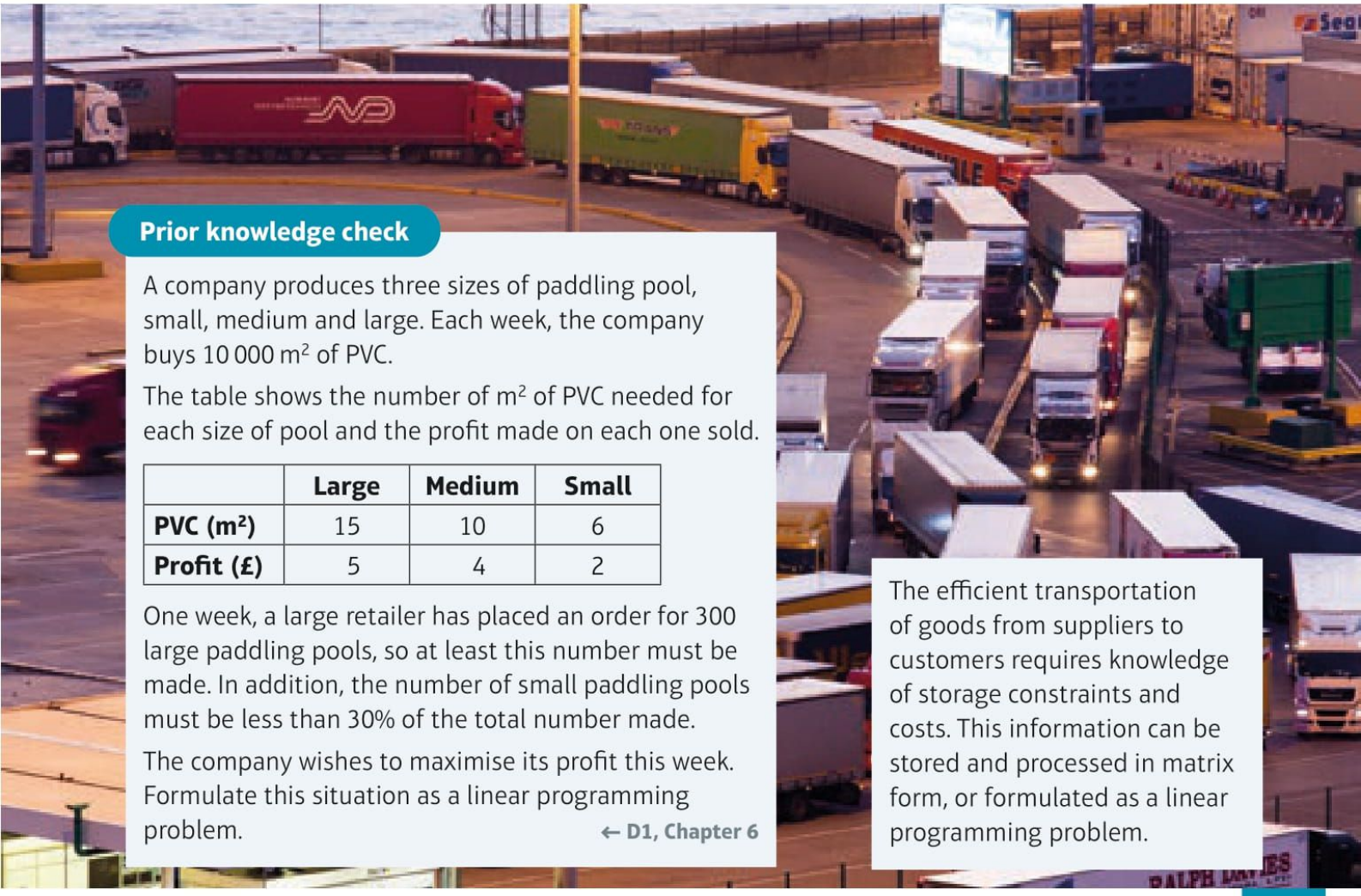
Transportation problems

1

Objectives

After completing this chapter you should be able to:

- Describe and model transportation problems → pages 2–37
- Use the north-west corner method → pages 2–6
- Understand unbalanced transportation problems and degenerate solutions → pages 6–11
- Use shadow costs to find improvement indices → pages 11–18
- Use the stepping-stone method → pages 19–28
- Formulate a transportation problem as a linear programming problem → pages 28–35



Prior knowledge check

A company produces three sizes of paddling pool, small, medium and large. Each week, the company buys 10 000 m^2 of PVC.

The table shows the number of m^2 of PVC needed for each size of pool and the profit made on each one sold.

	Large	Medium	Small
PVC (m^2)	15	10	6
Profit (£)	5	4	2

One week, a large retailer has placed an order for 300 large paddling pools, so at least this number must be made. In addition, the number of small paddling pools must be less than 30% of the total number made.

The company wishes to maximise its profit this week. Formulate this situation as a linear programming problem.

← D1, Chapter 6

The efficient transportation of goods from suppliers to customers requires knowledge of storage constraints and costs. This information can be stored and processed in matrix form, or formulated as a linear programming problem.

1.1 The north-west corner method

A

In this chapter you will look at the costs relating to the transportation of goods: to factories, and from factories to warehouses and customers. The problems considered are usually concerned with minimising distribution costs in situations where there are multiple sources and multiple destinations. You need to be familiar with the terminology used in describing and modelling transportation problems:

- **The capacity of each of the supply points (or sources) – the quantity of goods that can be produced at each factory or held at each warehouse. This is called the supply or stock.**
- **The amount required at each of the demand points (or destinations) – the quantity of goods that are needed at each shop or by each customer. This is called the demand.**
- **The unit cost of transporting goods from the supply points to the demand points.**

The unit cost is the cost of transporting one item. If the unit cost is £c then it will cost £nc to transport n items.

Example 1

Three suppliers A, B and C each produce road grit which has to be delivered to council depots W, X, Y and Z. The stock held at each supplier and the demand from each depot is known. The cost, in pounds, of transporting one lorry load of grit from each supplier to each depot is also known. This information is given in the table.

	Depot W	Depot X	Depot Y	Depot Z	Stock (lorry loads)
Supplier A	180	110	130	290	14
Supplier B	190	250	150	280	16
Supplier C	240	270	190	120	20
Demand (lorry loads)	11	15	14	10	50

This table is often referred to as the cost matrix.

This is the cost in pounds of transporting one lorry load from B to Y. This cost is sometimes written as C_{BY} or $C(BY)$.

Notice that the total supply is equal to the total demand. If this is not the case we introduce a dummy destination to absorb the excess supply, with transportation costs all zero. → Example 3

Use the information in the table to write down:

a the number of lorry loads of grit that each supplier can supply

b the number of lorry loads of grit required at each depot

c the cost of transporting a lorry load of grit from A to W

d the cost of transporting a lorry load of grit from C to Z.

e Which is the cheapest route to use?

f Which is the most expensive route to use?

a Suppliers A, B and C can provide 14, 16 and 20 lorry loads respectively.
b Depots W, X, Y and Z require 11, 15, 14 and 10 lorry loads respectively.
c The cost of transporting one lorry load from A to W is £180.
d The cost of transporting one lorry load from C to Z is £120.
e The cheapest route is A to X at £110 per load.
f The most expensive route is A to Z at £290 per load.

A In general, the manufacturer would like to minimise the total transportation costs whilst still meeting demand. This is called the **transportation problem** and can be solved using the **transportation algorithm**:

The transportation algorithm

1. First find an initial solution that uses all the stock and meets all the demands.
2. Calculate the total cost of this solution and see if it can be reduced by transporting some goods along a route not currently in the solution. (If this is not possible then the solution is optimal.)
3. If the cost can be reduced by using a new route, allocate as many units as possible to this new route to create a new solution.
4. Check the new solution in the same way as the initial solution to see if it is optimal. If not, repeat step 3.
5. When no further savings are possible, an optimal solution has been found.

The first step in the box above requires you to find an **initial solution**. The method used for this is called the **north-west corner method**:

Notation The solution generated by this method is sometimes called the initial (or basic) **feasible** solution.

The north-west corner method

1. Create a table, with one row for every source and one column for every destination. Each cell represents a route from a source to a destination. Each destination's demand is given at the foot of each column and each source's stock is given at the end of each row. Enter numbers in each cell to show how many units are to be sent along that route.
2. Begin with the top left-hand corner (the north-west corner). Allocate the maximum available quantity to meet the demand at this destination (whilst not exceeding the stock at this source).
3. As each stock is emptied, move one square down and allocate as many units as possible from the next source until the demand of the destination is met. As each demand is met, move one square to the right and again allocate as many units as possible.
4. When all the stock is assigned, and all the demands are met, stop.

Watch out In order to avoid degenerate solutions, movements are made between squares either vertically or horizontally but **never** diagonally. → Example 5

Note For a problem involving m sources and n destinations, you must enter $n + m - 1$ transportation quantities $\geqslant 0$. This will not reduce throughout the problem.

Note In your exam, problems will be restricted to a maximum of 4 sources and 4 destinations.

Example 2

A

	Depot W	Depot X	Depot Y	Depot Z	Stock
Supplier A	180	110	130	290	14
Supplier B	190	250	150	280	16
Supplier C	240	270	190	120	20
Demand	11	15	14	10	50

Use the north-west corner method to find an initial solution to the problem described in Example 1 and shown in the table, and state its cost.

Online Explore how the north-west corner method can be used to find an initial solution using GeoGebra.

Units transported

	W	X	Y	Z	Stock
A					14
B					16
C					20
Demand	11	15	14	10	50

1 Set up the table.

	W	X	Y	Z	Stock
A	11				14
B					16
C					20
Demand	11	15	14	10	50

2 Start to fill in the number of units you wish to send along each route, beginning at the north-west corner. Depot W requires 11 lorry loads. This does not exhaust the stock of supplier A.

	W	X	Y	Z	Stock
A	11	3			14
B					16
C					20
Demand	11	15	14	10	50

3 The demand at W has been met so move one square to the right and allocate $14 - 11 = 3$ units. The stock at A is now exhausted. The demand at X has not been met.

	W	X	Y	Z	Stock
A	11	3			14
B		12			16
C					20
Demand	11	15	14	10	50

As the stock at A has been exhausted, move one square down and allocate the maximum possible number of units from supplier B to depot X. In this case, $15 - 3 = 12$.

	W	X	Y	Z	Stock
A	11	3			14
B		12	4		16
C					20
Demand	11	15	14	10	50

Now that the demand at X has also been met, move one square to the right and use the remaining stock at B to start to meet the demand at Y.

A

	W	X	Y	Z	Stock
A	11	3			14
B		12	4		16
C			10		20
Demand	11	15	14	10	50

The stock at B is now exhausted (12 + 4 = 16) so move one square down and use the stock at C to fulfil the remaining demand at Y.

	W	X	Y	Z	Stock
A	11	3			14
B		12	4		16
C			10	10	20
Demand	11	15	14	10	50

Finally, move one square to the right and use the remaining stock at C to meet the demand at Z.

This is the final table. All of the stock has been used and all of the demands met.

Use this table, together with the table showing costs, to work out the total cost of the solution.

Cost matrix

	W	X	Y	Z	Stock
A	180	110	130	290	14
B	190	250	150	280	16
C	240	270	190	120	20
Demand	11	15	14	10	50

The total cost of this solution is

$(11 \times 180) + (3 \times 110) + (12 \times 250)$
$+ (4 \times 150) + (10 \times 190) + (10 \times 120)$
$= £9010$

Problem-solving

Number of occupied cells (routes used) in the table = number of supply points + number of demand points − 1.
In this case,
number of occupied cells (routes used) = 6
number of supply points = 3
number of demand points = 4
and 6 = 3 + 4 − 1.

Watch out The cost matrix shows the cost of transporting **one unit**. Multiply the entry in each relevant position by the number of units being transported by that route. The relevant entries are shaded in this table, as these are the routes used in this initial solution.

Exercise 1A

In questions 1 to 4, the tables show the unit costs of transporting goods from supply points to demand points, the number of units required at each demand point and the number of units available at each supply point. In each case:

a use the north-west corner method to find the initial solution

b verify that, for each solution,
number of occupied cells = number of supply points + number of demand points − 1

c determine the cost of each initial solution.

A 1

	P	Q	R	Supply
A	150	213	222	32
B	175	204	218	44
C	188	198	246	34
Demand	28	45	37	110

2

	P	Q	R	S	Supply
A	27	33	34	41	54
B	31	29	37	30	67
C	40	32	28	35	29
Demand	21	32	51	46	150

3

	P	Q	R	Supply
A	17	24	19	123
B	15	21	25	143
C	19	22	18	84
D	20	27	16	150
Demand	200	100	200	500

4

	P	Q	R	S	Supply
A	56	86	80	61	134
B	59	76	78	65	203
C	62	70	57	67	176
D	60	68	75	71	187
Demand	175	175	175	175	700

1.2 Unbalanced problems and degenerate solutions

In the cost matrices given in the previous section, the total supply was always equal to the total demand. In real life this is rarely the case.

- **When total supply $\neq$ total demand, a transportation problem is unbalanced.**
  - **If total supply > total demand, you need to add a dummy demand point.**
  - **If total supply < total demand, you need to add a dummy supply point.**

Hint If units are assigned to a dummy demand point, they represent excess capacity. If a particular demand point is receiving units from a dummy supply point, they represent unmet demand.

In each case the demand or supply at the dummy is chosen so that total demand is equal to total supply, and the transportation costs to/from the dummy location are all zero.

Example 3

Three outlets A, B and C are supplied by three suppliers X, Y and Z. The table shows the cost, in pounds, of transporting each unit, the number of units required at each outlet and the number of units available at each supplier.

	A	B	C	Supply
X	9	11	10	40
Y	10	8	12	60
Z	12	7	8	50
Demand	50	40	30	

a Explain why it is necessary to add a dummy demand point in order to solve this problem.

b Add a dummy demand point and appropriate costs to the table.

c Use the north-west corner method to obtain an initial solution.

A and the initial solution was

	W	*X*	*Y*	*Z*	Stock
A	11	3			14
B		12	4		16
C			10	10	20
Demand	11	15	14	10	50

The most negative improvement index was –130 in cell BW. ← Example 8

at a cost of £9010.

Obtain an improved solution and find the improved cost.

Use BW as the entering cell. This gave the most negative improvement index, so BW will be an increasing cell. Enter a value of θ into this cell.

θ is a non-negative value representing the number of units transported via the entering cell. You will later choose the largest possible value of θ that allows you to maintain the supply and demand constraints in other cells.

	W	*X*	*Y*	*Z*
A	11	3		
B	θ	12	4	
C			10	10

The adjustments can be made without keeping the Demand row and the Stock column.

In order to keep the demand at W correct, we must decrease the entry at AW, so AW will be a decreasing cell.

	W	*X*	*Y*	*Z*
A	$11 - \theta$	3		
B	θ	12	4	
C			10	10

In order to keep the stock at A correct, we must increase the entry at AX, so AX will be an increasing cell.

	W	*X*	*Y*	*Z*
A	$11 - \theta$	$3 + \theta$		
B	θ	12	4	
C			10	10

Online Explore how to obtain an improved solution with the stepping stone method using GeoGebra.



In order to keep the demand at X correct, we must decrease the entry at BX, so BX will be a decreasing cell.

	W	*X*	*Y*	*Z*
A	$11 - \theta$	$3 + \theta$		
B	θ	$12 - \theta$	4	
C			10	10

This is as far as you can go with adjustments since the top two rows both have an increasing and a decreasing cell.

A

Choose the largest possible value for θ that doesn't introduce any negative entries.

The largest possible value of θ is 11, so replace θ by 11 in the table:

	W	X	Y	Z
A	11 − 11	3 + 11		
B	11	12 − 11	4	
C			10	10

This gives the following improved solution.

	W	X	Y	Z
A		14		
B	11	1	4	
C			10	10

Looking at the table of transportation costs, this solution has a cost of £7580.

Note All of the stages are shown here in separate tables to make the method clear but, in your exam, you can carry out all of the steps in a single table.

Watch out The table needs to show a feasible solution, so you can't have any negative entries. But the choice of θ will always reduce at least one other entry to 0.

Problem-solving

As a double check, it is always true that

New cost = cost of former solution + improvement index × θ

In this case, 7580 = 9010 + (−130) × 11

In the above example, the choice of θ means that the number of units being transported via AW is 0. Your solution needs to have exactly $n + m - 1$ entries in the table, so don't write 0 in AW. You should **leave it blank** instead.

- **One entry that has been reduced to 0 by the stepping-stone method is called the exiting cell, and should be left blank in the improved solution.**
- **At each iteration you create one entering cell and one exiting cell.**
- **To find an optimal solution, continue to calculate new shadow costs and improvement indices and then apply the stepping-stone method. Repeat this iteration until all the improvement indices are non-negative.**

Watch out If two cells are reduced to 0 then choose one as the exiting cell, and leave the other 0 in the table. You should always check that your solution has $n + m - 1$ entries.

Example 11

Find an optimal solution for Example 10.

Second iteration

Find the new shadow costs:

		50	110	10	−60
		W	X	Y	Z
0	A		110		
140	B	190	250	150	
180	C			190	120

Challenge

A The table shows the transportation costs, in pounds, for each unit of stock from three warehouses A, B and C to three retailers P, Q and R. It also shows the availability of stock at the warehouses, and the demand at each of the retailers, over a period of one week.

	P	Q	R	Stock
A	7	8	6	14
B	5	7	9	12
C	6	8	8	16
Demand	15	9	11	

Any stock that remains in a warehouse at the end of the week incurs a storage charge. The weekly storage charges for each unit of surplus stock are £3 at warehouse A, £5 at warehouse B and £4 at warehouse C.
The total weekly cost, including storage charges, is to be minimised.

a Add a suitably labelled dummy to the table to represent the problem.

b Formulate the problem as a linear programming problem.

Summary of key points

1 You need to be familiar with the following terminology:

- The capacity of each of the **supply points** (or **sources**) – the quantity of goods that can be produced at each factory or held at each warehouse. This is called the **supply** or **stock**.
- The amount required at each of the **demand points** (or **destinations**) – the quantity of goods that are needed at each shop or by each customer. This is called the **demand**.
- The **unit cost** of transporting goods from the supply points to the demand points.

2 The steps in the **transportation algorithm** are as follows:

- First find an initial solution that uses all the stock and meets all the demands.
- Calculate the total cost of this solution and see if it can be reduced by transporting some goods along a route not currently in the solution. (If this is not possible then the solution is optimal.)
- If the cost can be reduced by using a new route, allocate as many units as possible to this new route to create a new solution.
- Check the new solution in the same way as the initial solution to see if it is optimal. If not, repeat the previous step.
- When no further savings are possible, an optimal solution has been found.

3 The **north-west corner method**:

- Create a table, with one row for every source and one column for every destination. Each cell represents a route from a source to a destination. Each destination's demand is given at the foot of each column and each source's stock is given at the end of each row. Enter numbers in each cell to show how many units are to be sent along that route.

Pearson Edexcel AS and A level Further Mathematics

Decision Mathematics 2

Series Editor: Harry Smith

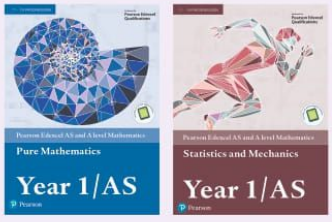
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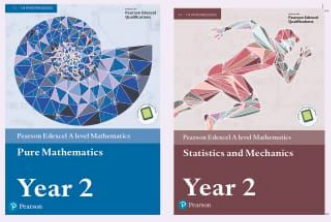
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Year 1/AS

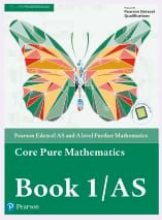


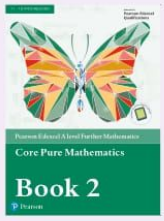
Year 2



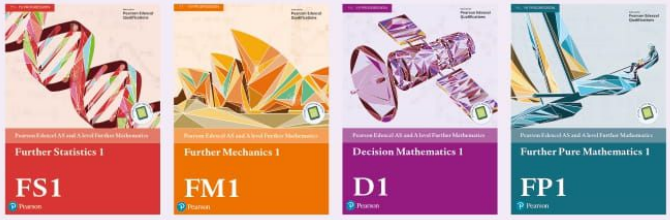
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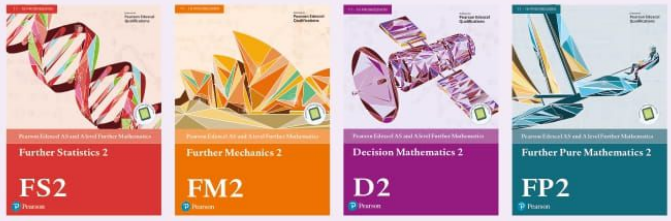
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