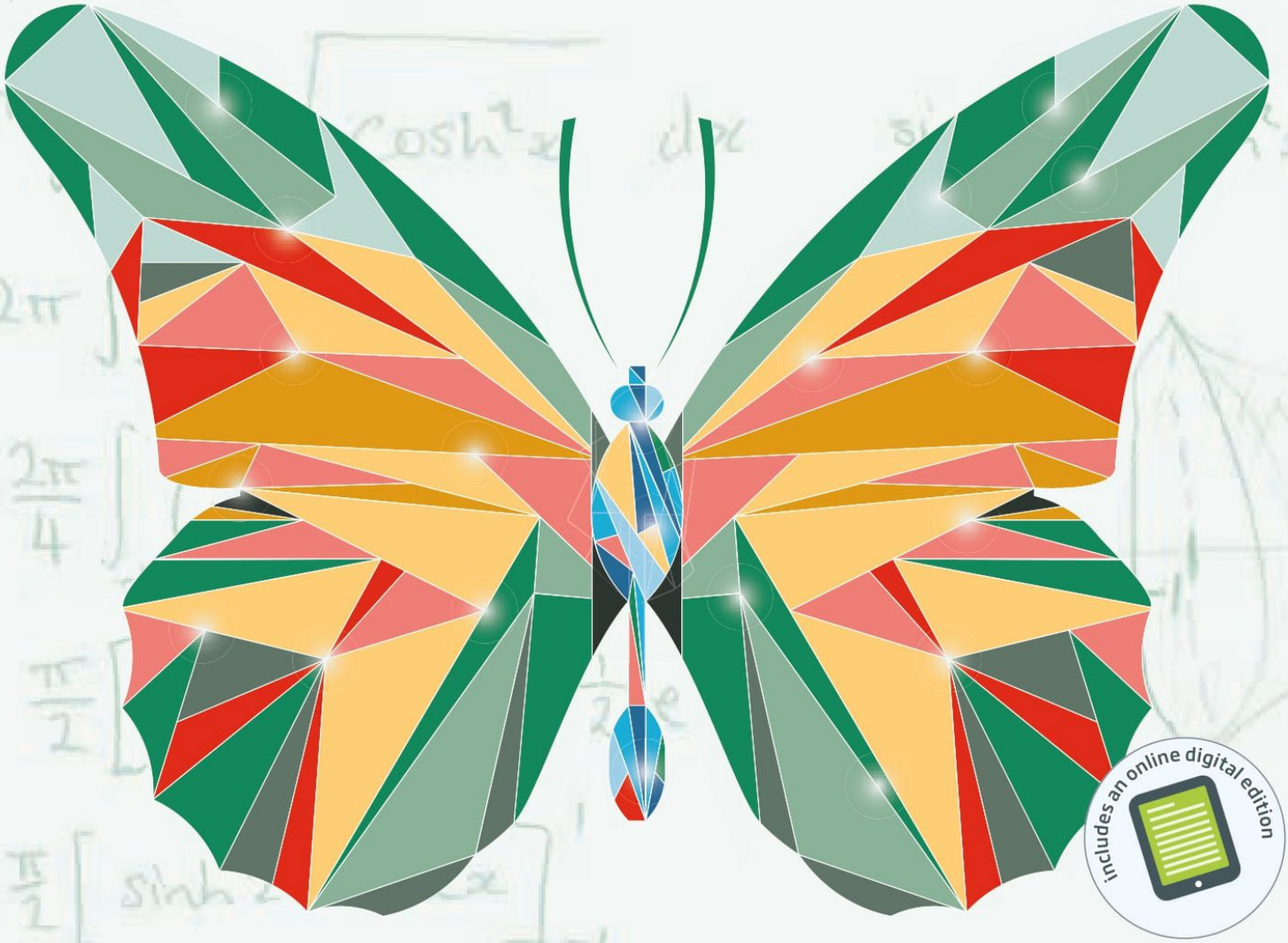
11 - 19 PROGRESSION

Endorsed for
Pearson Edexcel Qualifications



Pearson Edexcel A level Further Mathematics

Core Pure Mathematics

Book 2

Pearson

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Overarching themes

The following three overarching themes have been fully integrated throughout the Pearson Edexcel AS and A level Mathematics series, so they can be applied alongside your learning and practice.

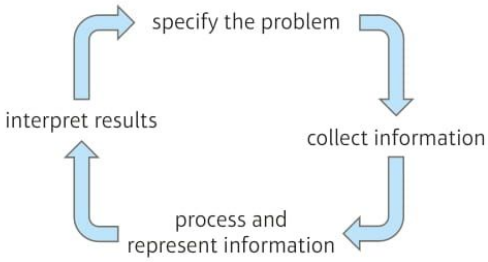
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols
- Dedicated sections on mathematical proof explain key principles and strategies
- Opportunities to critique arguments and justify methods

2. Mathematical problem solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Structured and unstructured questions to build confidence
- Challenge boxes provide extra stretch

The Mathematical Problem-solving cycle

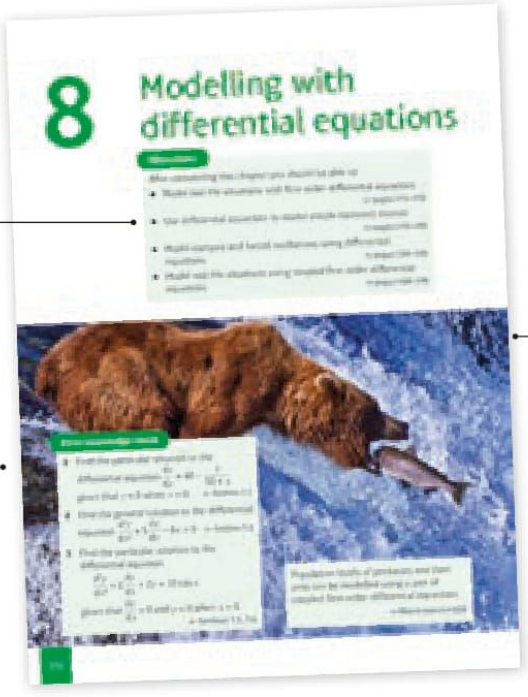


3. Mathematical modelling

- Dedicated modelling sections in relevant topics provide plenty of practice where you need it
- Examples and exercises include qualitative questions that allow you to interpret answers in the context of the model
- Dedicated chapter in Statistics & Mechanics Year 1/AS explains the principles of modelling in mechanics

Finding your way around the book

Access an online digital edition using the code at the front of the book.



Each chapter starts with a list of objectives

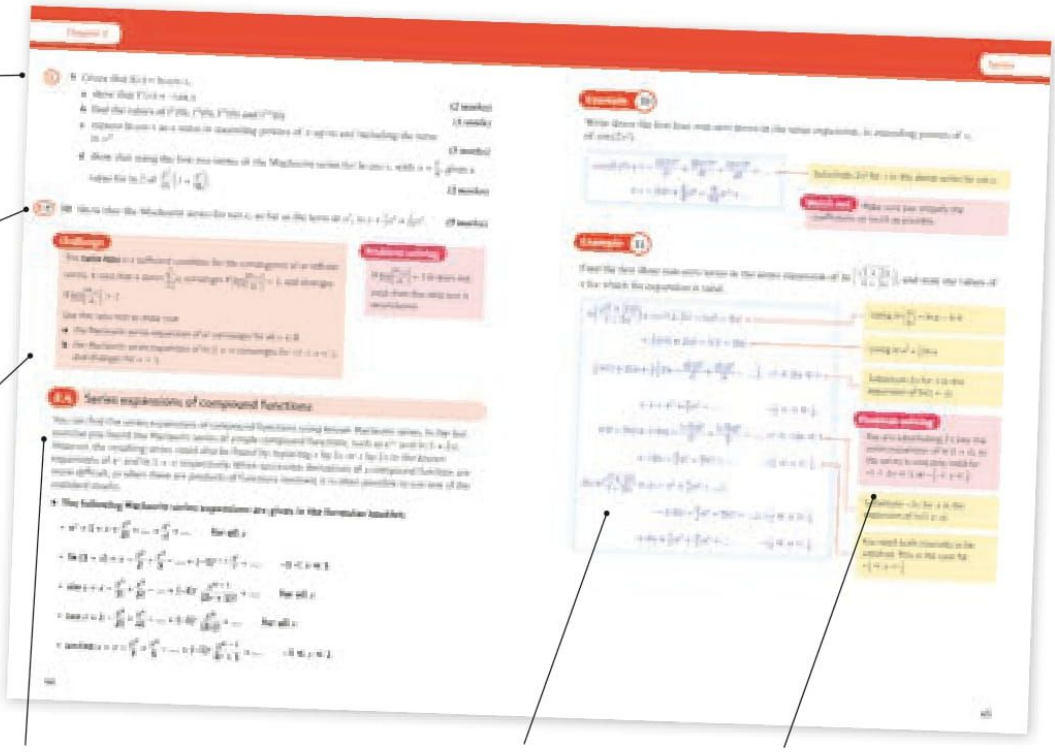
The real world applications of the maths you are about to learn are highlighted at the start of the chapter with links to relevant questions in the chapter

The *Prior knowledge check* helps make sure you are ready to start the chapter

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams

Challenge boxes give you a chance to tackle some more difficult questions



Exam-style questions are flagged with (E)

Problem-solving questions are flagged with (P)

Each section begins with explanation and key learning points

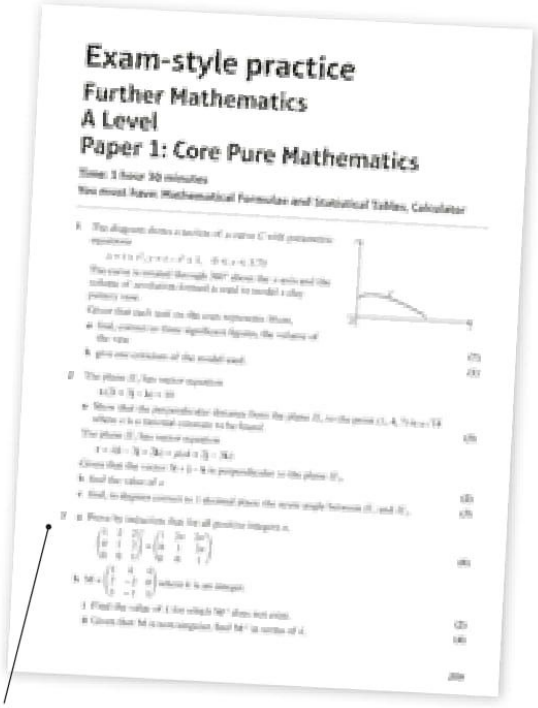
Each chapter ends with a *Mixed exercise* and a *Summary of key points*

Step-by-step worked examples focus on the key types of questions you'll need to tackle

Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

Every few chapters a *Review exercise* helps you consolidate your learning with lots of exam-style questions





Two A level practice papers at the back of the book help you prepare for the real thing.

Extra online content

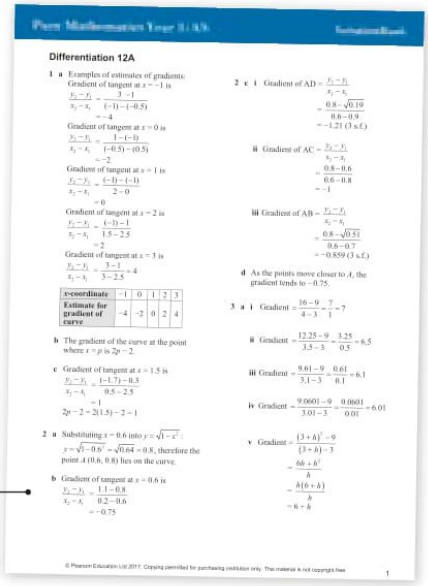
Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

SolutionBank provides a full worked solution for every question in the book.

Online Full worked solutions are available in SolutionBank.

Download all the solutions as a PDF or quickly find the solution you need online


Pure Mathematics Year 1 / AS

Differentiation 12A

1 a Examples of estimates of gradients:
Gradient of tangent at $x = -1$ is
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{(-1) - (-0.5)} = -4$
Gradient of tangent at $x = 0$ is
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-1)}{(-0.5) - (0.5)} = -2$
Gradient of tangent at $x = 1$ is
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - (-1)}{2 - 0} = 0$
Gradient of tangent at $x = 2$ is
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - 1}{1.5 - 2.5} = 2$
Gradient of tangent at $x = 3$ is
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{3 - 2.5} = 4$

x-coordinate	−1	0	1	2	3
Estimate for gradient of curve	−4	−2	0	2	4

b The gradient of the curve at the point where $x = p$ is $2p - 2$.

c Gradient of tangent at $x = 1.5$ is
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1.7) - 0.3}{0.5 - 2.5} = 1$
$2p - 2 = 2(1.5) - 2 = 1$

2 a Substituting $x = 0.6$ into $y = \sqrt{1 - x^2}$:
$y = \sqrt{1 - 0.6^2} = \sqrt{0.64} = 0.8$, therefore the point $A\,(0.6, 0.8)$ lies on the curve.

b Gradient of tangent at $x = 0.6$ is
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1.1 - 0.8}{0.2 - 0.6} = -0.75$

2 c i Gradient of AD $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.8 - \sqrt{0.19}}{0.6 - 0.9} = -1.21\ (3\text{ s.f.})$

ii Gradient of AC $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.8 - 0.6}{0.6 - 0.8} = -1$

iii Gradient of AB $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.8 - \sqrt{0.51}}{0.6 - 0.7} = -0.859\ (3\text{ s.f.})$

d As the points move closer to A, the gradient tends to -0.75.

3 a i Gradient $= \frac{16 - 9}{4 - 3} = \frac{7}{1} = 7$

ii Gradient $= \frac{12.25 - 9}{3.5 - 3} = \frac{3.25}{0.5} = 6.5$

iii Gradient $= \frac{9.61 - 9}{3.1 - 3} = \frac{0.61}{0.1} = 6.1$

iv Gradient $= \frac{9.0601 - 9}{3.01 - 3} = \frac{0.0601}{0.01} = 6.01$

v Gradient $= \frac{(3 + h)^2 - 9}{(3 + h) - 3} = \frac{6h + h^2}{h} = \frac{h(6 + h)}{h} = 6 + h$

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1

Use of technology

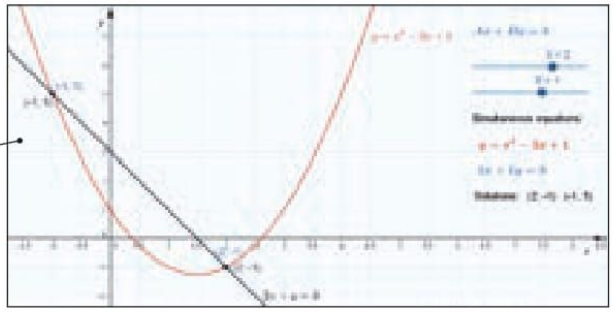
Explore topics in more detail, visualise problems and consolidate your understanding using pre-made GeoGebra activities.

Online Find the point of intersection graphically using technology.

GeoGebra

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Interact with the maths you are learning using GeoGebra's easy-to-use tools



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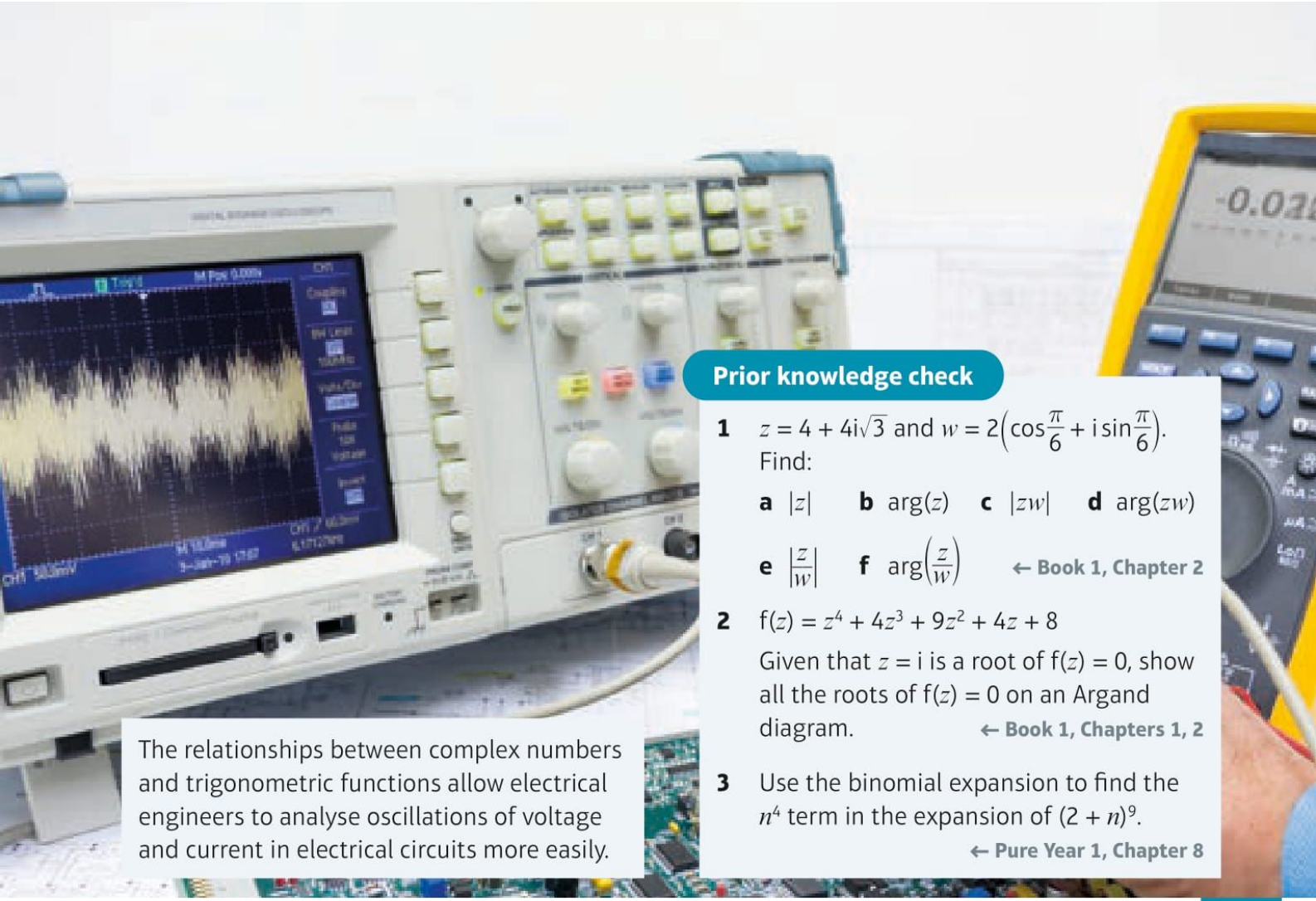
Complex numbers

1

Objectives

After completing this chapter you should be able to:

- Express a complex number in exponential form → pages 2–5
- Multiply and divide complex numbers in exponential form → pages 5–8
- Understand de Moivre's theorem → pages 8–11
- Use de Moivre's theorem to derive trigonometric identities → pages 11–15
- Use de Moivre's theorem to find sums of series → pages 16–19
- Know how to solve completely equations of the form $z^n - a - ib = 0$, giving special attention to cases where $a = 1$ and $b = 0$ → pages 20–25
- Use complex roots of unity to solve geometric problems → pages 25–27



The relationships between complex numbers and trigonometric functions allow electrical engineers to analyse oscillations of voltage and current in electrical circuits more easily.

Prior knowledge check

1 $z = 4 + 4i\sqrt{3}$ and $w = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.
Find:

a $|z|$ **b** $\arg(z)$ **c** $|zw|$ **d** $\arg(zw)$

e $\left|\frac{z}{w}\right|$ **f** $\arg\left(\frac{z}{w}\right)$ ← Book 1, Chapter 2

2 $f(z) = z^4 + 4z^3 + 9z^2 + 4z + 8$

Given that $z = i$ is a root of $f(z) = 0$, show all the roots of $f(z) = 0$ on an Argand diagram. ← Book 1, Chapters 1, 2

3 Use the binomial expansion to find the n^4 term in the expansion of $(2 + n)^9$.
← Pure Year 1, Chapter 8

1.1 Exponential form of complex numbers

You can use the modulus–argument form of a complex number to express it in the **exponential form**: $z = re^{i\theta}$.

Links The **modulus–argument** form of a complex number is $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$.
← Book 1, Section 2.3

You can write $\cos\theta$ and $\sin\theta$ as infinite series of powers of θ:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + \frac{(-1)^r\,\theta^{2r}}{(2r)!} + \ldots \quad (1)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots + \frac{(-1)^r\,\theta^{2r+1}}{(2r+1)!} + \ldots \quad (2)$$

You can also write e^x, $x \in \mathbb{R}$, as a series expansion in powers of x.

Links These are the Maclaurin series expansions of $\sin\theta$, $\cos\theta$ and e^x.
→ Chapter 2

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^r}{r!} + \ldots$$

You can use this expansion to define the exponential function for complex powers, by replacing x with a complex number. In particular, if you replace x with the imaginary number $i\theta$, you get

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

By comparing this series expansion with (1) and (2), you can write $e^{i\theta}$ as

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This formula is known as **Euler's relation**. It is important for you to remember this result.

- **You can use Euler's relation, $e^{i\theta} = \cos\theta + i\sin\theta$, to write a complex number z in exponential form:**

  $$z = re^{i\theta}$$

  where $r = |z|$ and $\theta = \arg z$.

Note Substituting $\theta = \pi$ into Euler's relation yields **Euler's identity**:

$$e^{i\pi} + 1 = 0$$

This equation links the five fundamental constants 0, 1, π, e and i, and is considered an example of mathematical beauty.

Example 1

Express the following in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$ **b** $z = 5\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

Compare with $r(\cos\theta + i\sin\theta)$.

So $r = \sqrt{2}$ and $\theta = \frac{\pi}{10}$

Therefore, $z = \sqrt{2}e^{\frac{\pi i}{10}}$

$z = re^{i\theta}$

b $z = 5\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

$z = 5\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right)$

Problem-solving

Use $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

So $r = 5$ and $\theta = -\frac{\pi}{8}$

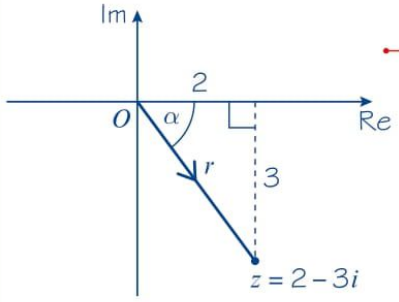
Compare with $r(\cos\theta + i\sin\theta)$.

Therefore, $z = 5e^{-\frac{\pi i}{8}}$

$z = re^{i\theta}$.

Example 2

Express $z = 2 - 3i$ in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the complex number.

Here z is in the fourth quadrant so the required argument is $-\alpha$.

$r = |z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\theta = \arg z = -\arctan\left(\frac{3}{2}\right) = -0.983$ (3 s.f.)

Find r and θ.

Therefore, $z = \sqrt{13}e^{-0.983i}$

$z = re^{i\theta}$.

Example 3

Express $z = \sqrt{2}e^{\frac{3\pi i}{4}}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

$z = \sqrt{2}e^{\frac{3\pi i}{4}}$, so $r = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$ — Compare with $re^{i\theta}$.

$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ — Write z in modulus–argument form.

$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

Therefore, $z = -1 + i$ — Simplify.

Example 4

Express $z = 2e^{\frac{23\pi i}{5}}$ in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

$z = 2e^{\frac{23\pi i}{5}}$, so $r = 2$ and $\theta = \frac{23\pi}{5}$ — Compare with $re^{i\theta}$.

$\frac{23\pi}{5} - 2\pi = \frac{13\pi}{5}$, $\frac{13\pi}{5} - 2\pi = \frac{3\pi}{5}$

$\frac{3\pi}{5}$ is in the range $-\pi < \theta \leqslant \pi$

So $z = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$ — Write z in the form $r(\cos\theta + i\sin\theta)$.

Problem-solving

$\cos\theta = \cos(\theta + 2\pi)$ and $\sin\theta = \sin(\theta + 2\pi)$.

Subtract multiples of 2π from $\frac{23\pi}{5}$ until you find a value in the range $-\pi < \theta \leqslant \pi$.

Example 5

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

$e^{i\theta} = \cos\theta + i\sin\theta$ (1)

$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta)$

So $e^{-i\theta} = \cos\theta - i\sin\theta$ (2) — Use $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$ — Add (1) and (2).

$\Rightarrow \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$ — Divide both sides by 2.

Hence, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, as required.

Exercise 1A

1 Express the following in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$. Use exact values of r and θ where possible, or values to 3 significant figures otherwise.

a -3 **b** $6i$ **c** $-2\sqrt{3} - 2i$

d $-8 + i$ **e** $2 - 5i$ **f** $-2\sqrt{3} + 2i\sqrt{3}$

g $\sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ **h** $8\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$ **i** $2\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$

2 Express the following in the form $x + iy$ where $x, y \in \mathbb{R}$.

a $e^{\frac{\pi i}{3}}$ **b** $4e^{\pi i}$ **c** $3\sqrt{2}\,e^{\frac{\pi i}{4}}$

d $8e^{\frac{\pi i}{6}}$ **e** $3e^{-\frac{\pi i}{2}}$ **f** $e^{\frac{5\pi i}{6}}$

g $e^{-\pi i}$ **h** $3\sqrt{2}e^{-\frac{3\pi i}{4}}$ **i** $8e^{-\frac{4\pi i}{3}}$

3 Express the following in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

a $e^{\frac{16\pi i}{13}}$ **b** $4e^{\frac{17\pi i}{5}}$ **c** $5e^{-\frac{9\pi i}{8}}$

(P) **4** Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

1.2 Multiplying and dividing complex numbers

You can apply the modulus–argument rules for multiplying and dividing complex numbers to numbers written in exponential form.

Recall that, for any two complex numbers z_1 and z_2,

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Links These results can be proved by considering the numbers z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$ and using the addition formulae for cos and sin. ← Book 1, Section 2.3

Applying these results to numbers in exponential form gives the following result:

■ **If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then:**

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Watch out You cannot automatically assume the laws of indices work the same way with complex numbers as with real numbers. This result only shows that they can be applied in these specific cases.

Example 6

a Express $2e^{\frac{\pi i}{6}} \times \sqrt{3}e^{\frac{\pi i}{3}}$ in the form $x + iy$.

b $z = 2 + 2i$, $\text{Im}(zw) = 0$ and $|zw| = 3|z|$
Use geometrical reasoning to find the two possibilities for w, giving your answers in exponential form.

a $2e^{\frac{\pi i}{6}} \times \sqrt{3}\, e^{\frac{\pi i}{3}} = (2 \times \sqrt{3})e^{i\left(\frac{\pi}{6} + \frac{\pi}{3}\right)}$

$= 2\sqrt{3}\, e^{\frac{\pi i}{2}}$

$= 2\sqrt{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= 2\sqrt{3}(0 + i)$

$= 2i\sqrt{3}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

Simplify.

Convert the complex number to modulus–argument form.

b $|zw| = 3|z| \Rightarrow |w| = 3$

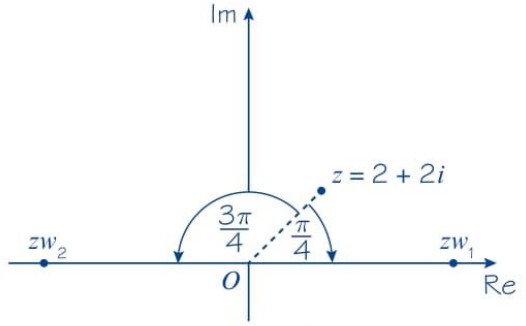
$|zw| = |z||w| = 3|z|$.

$\arg z = \arctan\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$\text{Im}(zw) = 0$ so $\arg(zw) = 0$ or π

So $\arg w = \frac{3\pi}{4}$ or $-\frac{\pi}{4}$

wz lies on the real axis, so z is rotated $\frac{3\pi}{4}$ clockwise or $\frac{\pi}{4}$ anticlockwise when multiplied by w.



$w_1 = 3e^{-\frac{\pi i}{4}}$ and $w_2 = 3e^{\frac{3\pi i}{4}}$

Example 7

Express $\dfrac{2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$ in the form $re^{i\theta}$.

$\dfrac{2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)} = \dfrac{2e^{\frac{\pi i}{12}}}{\sqrt{2}e^{\frac{5\pi i}{6}}}$

$= \dfrac{2}{\sqrt{2}}e^{i\left(\frac{\pi}{12} - \frac{5\pi}{6}\right)}$

$= \sqrt{2}\, e^{-\frac{3\pi i}{4}}$

Convert the numerator and denominator to exponential form.

$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}e^{i(\theta_1 - \theta_2)}$

Simplify.

Exercise 1B

1 Express the following in the form $x + iy$, where $x, y \in \mathbb{R}$.

a $e^{\frac{\pi i}{3}} \times e^{\frac{\pi i}{4}}$ **b** $\sqrt{5}e^{i\theta} \times 3e^{3i\theta}$ **c** $\sqrt{2}e^{\frac{2\pi i}{3}} \times e^{-\frac{7\pi i}{3}} \times 3e^{\frac{\pi i}{6}}$

2 Express the following in the form $x + iy$ where $x, y \in \mathbb{R}$.

a $\dfrac{2e^{\frac{7\pi i}{2}}}{8e^{\frac{9\pi i}{2}}}$ **b** $\dfrac{\sqrt{3}e^{\frac{3\pi i}{7}}}{4e^{-\frac{2\pi i}{7}}}$ **c** $\dfrac{\sqrt{2}e^{-\frac{15\pi i}{6}}}{2e^{\frac{\pi i}{3}}} \times \sqrt{2}e^{\frac{19\pi i}{3}}$

3 Express the following in the form $re^{i\theta}$.

a $(\cos 2\theta + i\sin 2\theta)(\cos 3\theta + i\sin 3\theta)$ **b** $\left(\cos\frac{3\pi}{11} + i\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11} + i\sin\frac{8\pi}{11}\right)$

c $3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \times 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

d $\sqrt{6}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right) \times \sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

4 Express the following in the form $re^{i\theta}$.

a $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta + i\sin 2\theta}$ **b** $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$ **c** $\dfrac{3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$

5 z and w are two complex numbers where $z = -9 + 3i\sqrt{3}$, $|w| = \sqrt{3}$ and $\arg w = \dfrac{7\pi}{12}$

Express the following in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.

a z **b** w **c** zw **d** $\dfrac{z}{w}$

(P) **6** Use the exponential form for a complex number to show that

$$\frac{(\cos 9\theta + i\sin 9\theta)(\cos 4\theta + i\sin 4\theta)}{\cos 7\theta + i\sin 7\theta} \equiv \cos 6\theta + i\sin 6\theta$$

(E/P) **7** $z = 1 + i\sqrt{3}$, $\mathrm{Re}\left(\dfrac{z^2}{w}\right) = 0$ and $\left|\dfrac{z^2}{w}\right| = |z|$

Use geometrical reasoning to find the two possibilities for w, giving your answers in exponential form. **(4 marks)**

(E/P) **8** **a** Evaluate $(1 + i)^2$, giving your answer in exponential form. **(2 marks)**

b Use mathematical induction to prove that $(1 + i)^n = 2^{\frac{n}{2}}e^{\frac{n\pi i}{4}}$ for $n \in \mathbb{Z}^+$. **(4 marks)**

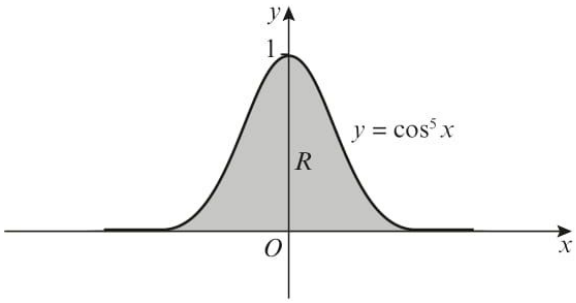
c Hence find $(1 + i)^{16}$. **(1 mark)**

(P) **9** Use Euler's relation for $e^{i\theta}$ and $e^{-i\theta}$ to verify that $\cos^2\theta + \sin^2\theta \equiv 1$.

E/P **6 a** Show that

$$\cos^5\theta \equiv \tfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$ **(5 marks)**

The diagram shows the curve with equation $y = \cos^5 x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$. The finite region R is bounded by the curve and the x-axis.



b Calculate the exact area of R. **(6 marks)**

E/P **7 a** Show that

$$\sin^6\theta \equiv -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$$ **(5 marks)**

b Using the substitution $\alpha = \left(\frac{\pi}{2} - \theta\right)$, or otherwise, find a similar identity for $\cos^6\theta$. **(3 marks)**

c Given that $\int_0^a \cos^6\theta + \sin^6\theta\,\mathrm{d}\theta = \frac{5\pi}{32}$, find the exact value of a. **(5 marks)**

E/P **8** Use de Moivre's theorem to show that

$$\sin 6\theta \equiv \sin 2\theta(16\cos^4\theta - 16\cos^2\theta + 3)$$ **(5 marks)**

E/P **9 a** Use de Moivre's theorem to show that

$$\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$ **(5 marks)**

b Hence find all solutions to the equation

$$16x^5 - 20x^3 + 5x + 1 = 0$$

giving your answers to 3 decimal places where necessary. **(5 marks)**

E/P **10 a** Show that

$$\sin^5\theta \equiv \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$ **(5 marks)**

b Hence solve the equation

$$\sin 5\theta - 5\sin 3\theta + 9\sin\theta = 0 \text{ for } 0 \leqslant \theta < \pi$$ **(4 marks)**

E/P **11 a** Use de Moivre's theorem to show that $\cos 5\theta \equiv \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$ **(5 marks)**

b By solving the equation $\cos 5\theta = 0$, deduce that $\cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$ **(4 marks)**

c Hence, or otherwise, write down the exact values of $\cos^2\left(\frac{3\pi}{10}\right)$, $\cos^2\left(\frac{7\pi}{10}\right)$ and $\cos^2\left(\frac{9\pi}{10}\right)$. **(3 marks)**

E/P **12** **a** Use de Moivre's theorem to find an expression for $\tan 3\theta$ in terms of $\tan\theta$. **(4 marks)**

b Deduce that $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$ **(2 marks)**

E/P **13** The infinite series C and S are defined as

$$C = 1 + k\cos\theta + k^2\cos 2\theta + k^3\cos 3\theta + \ldots$$

$$S = k\sin\theta + k^2\sin 2\theta + k^3\sin 3\theta + \ldots$$

where k is a real number and $|k| < 1$.

By considering $C + \mathrm{i}S$, show that $C = \dfrac{1 - k\cos\theta}{1 + k^2 - 2k\cos\theta}$ and write down the corresponding expression for S. **(8 marks)**

E **14** **a** Express $4 - 4\mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$, $-\pi < \theta \leqslant \pi$, where r and θ are exact values. **(2 marks)**

b Hence, or otherwise, solve the equation $z^5 = 4 - 4\mathrm{i}$, leaving your answers in the form $z = Re^{ik\pi}$, where R is the modulus of z and k is a rational number such that $-1 \leqslant k \leqslant 1$. **(4 marks)**

c Show on an Argand diagram the points representing the roots. **(2 marks)**

E/P **15** **a** Find the cube roots of $2 - 2\mathrm{i}$ in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. **(5 marks)**

These cube roots are represented by points A, B and C in the Argand diagram, with A in the fourth quadrant and ABC going anticlockwise. The midpoint of AB is M, and M represents the complex number w.

b Draw an Argand diagram, showing the points A, B, C and M. **(2 marks)**

c Find the modulus and argument of w. **(2 marks)**

d Find w^6 in the form $a + b\mathrm{i}$. **(3 marks)**

E/P **16** An equilateral triangle has its centre at the origin and one vertex at the point (2, 1).

a Find the coordinates of the other two vertices. **(4 marks)**

b Show that the length of one side of the triangle is $\sqrt{15}$. **(2 marks)**

Challenge

Show that the points on an Argand diagram that represent the roots of $\left(\dfrac{z+1}{z}\right)^6 = 1$ lie on a straight line.

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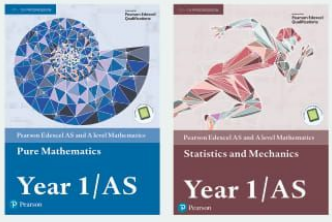
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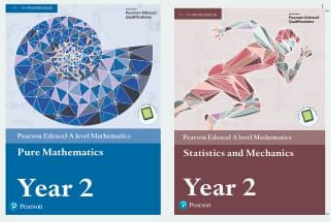
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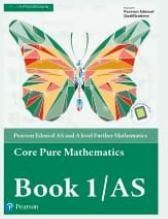


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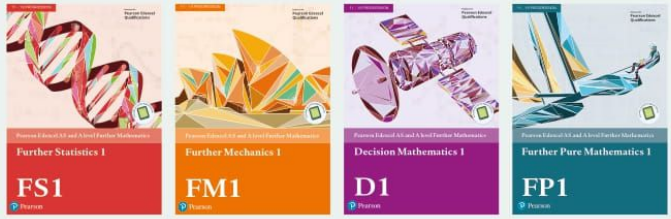
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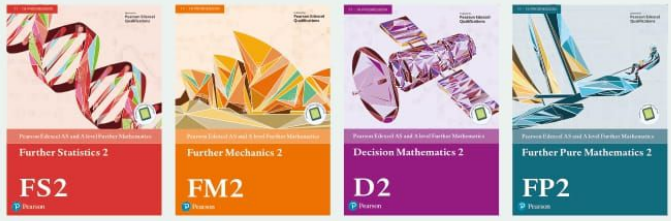
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