CHAPTER 3

C3: Inverse trigonometric functions and secant, cosecant and cotangent

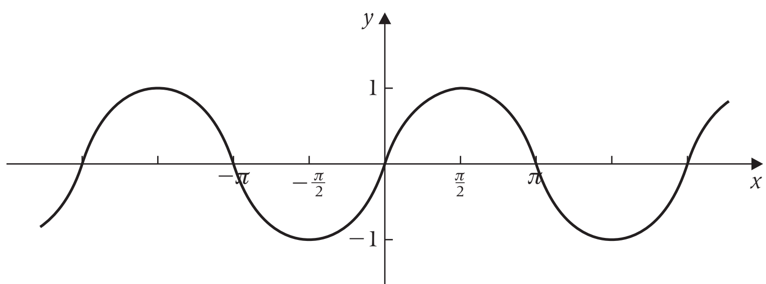
Learning objectives

After studying this chapter, you should be able to:

- work with the inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ and be able to draw their graphs over appropriate restricted domains
- understand the secant, cosecant and cotangent functions
- sketch the graphs of the secant, cosecant and cotangent functions
- know and use the two identities relating to the squares of the secant, cosecant and cotangent functions to prove other identities and to solve equations.

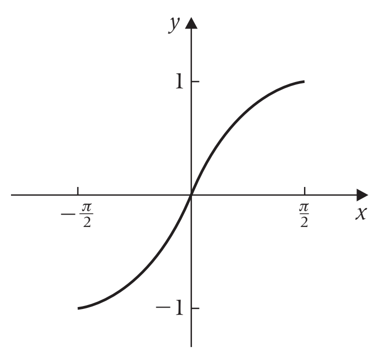
3.1 Inverse trigonometric functions

In section 1.11 you read that in order for an inverse function f^{-1} to exist, the function f must be one-one. The sine function, defined over the domain of real numbers, is clearly a many-one function as can be seen from its graph.



$y = \sin x$

So, at first sight, it would seem that the sine function has no inverse. To overcome this problem the domain of the sine function is restricted so that it becomes a one-one function but still takes all real values in the range $-1 \leqslant \sin x \leqslant 1$. You can do this by restricting the domain to $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, where x is in radians.



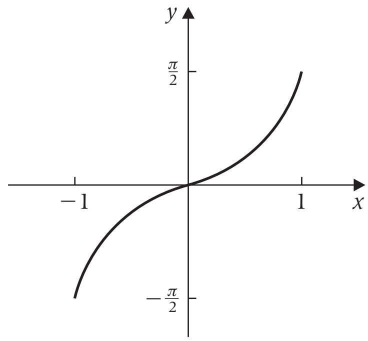
The inverse function of $\sin x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, is written as $\sin^{-1} x$.

From the general definition of an inverse function you can deduce that $y = \sin^{-1} x \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.

$\sin^{-1} x$ has domain $-1 \leqslant x \leqslant 1$ and range $-\dfrac{\pi}{2} \leqslant \sin^{-1} x \leqslant \dfrac{\pi}{2}$.

The graph of $y = \sin^{-1} x$, $-1 \leqslant x \leqslant 1$ is obtained by reflecting the graph of $y = \sin x$, $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$, in the line $y = x$.

Warning: $\sin^{-1} x \neq \dfrac{1}{\sin x}$.



Worked example 3.1

Find the exact values of:

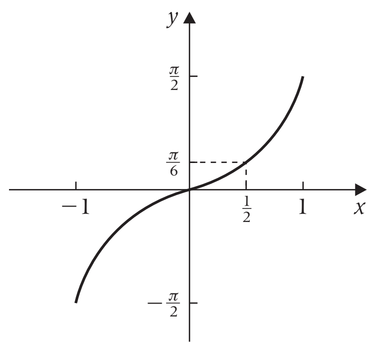
(a) $\sin^{-1} \dfrac{1}{2}$

(b) $\cos\left(\sin^{-1} \dfrac{1}{3}\right)$

Solution

(a) Since an exact answer is required, using your calculator set in degree mode, $\sin^{-1} \dfrac{1}{2} = 30°$, and $30° = \dfrac{\pi}{180} \times 30 = \dfrac{\pi}{6}$ radians.

So $\sin^{-1} \dfrac{1}{2} = \dfrac{\pi}{6}$, since $\sin \dfrac{\pi}{6} = \dfrac{1}{2}$ and $\dfrac{\pi}{6}$ lies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

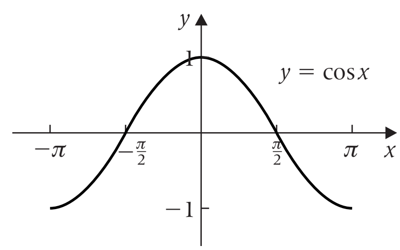


(b) Let $\alpha = \sin^{-1} \dfrac{1}{3} \Rightarrow \sin \alpha = \dfrac{1}{3}$.

Using the identity $\cos^2 \alpha + \sin^2 \alpha = 1$, you have $\cos^2 \alpha = \dfrac{8}{9}$.

Since α has to lie between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ you can deduce that $\cos \alpha$ is positive.

$$\Rightarrow \cos(\sin^{-1} \tfrac{1}{3}) = \cos \alpha = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}.$$



Although the following direct results are true

$\sin^{-1}(\sin x) = x$ for $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$,
$\sin(\sin^{-1} x) = x$ for $-1 \leqslant x \leqslant 1$.

Composite functions and inverse functions are considered in chapter 1.

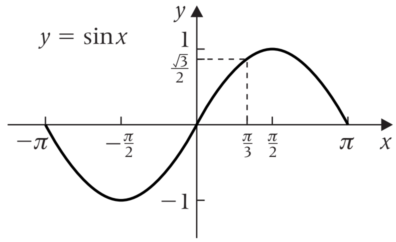
you must be careful when x lies outside the given inequalities. The next worked example shows you how to deal with such a case.

Worked example 3.2

Given that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, find the exact value of $\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$.

Solution

Since $\frac{4\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ you cannot use the direct result. Instead you evaluate



$\sin\frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$.

Let $\alpha = \sin^{-1}\left(\sin\frac{4\pi}{3}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \Rightarrow \sin\alpha = -\frac{\sqrt{3}}{2}$.

The value of α which satisfies $\sin\alpha = -\frac{\sqrt{3}}{2}$ and lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is $-\frac{\pi}{3}$

$\Rightarrow \sin^{-1}\left(\sin\frac{4\pi}{3}\right) = -\frac{\pi}{3}$.

The inverse functions of $\cos x$ and $\tan x$ are dealt with in a similar way.

The restricted domain for $\cos x$ is not the same as that used for $\sin x$.

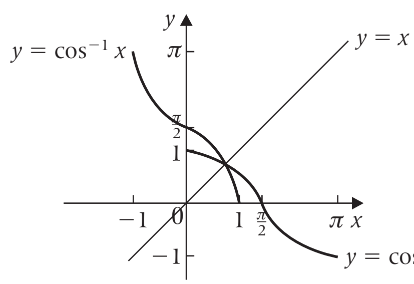
Restricting the domain of $\cos x$ to $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ would not give the negative values in the range 0 to −1.

The inverse function of $\cos x$, $0 \leqslant x \leqslant \pi$, is written as $\cos^{-1} x$.
$y = \cos^{-1} x \Leftrightarrow \cos y = x$ and $0 \leqslant y \leqslant \pi$.

$\cos^{-1} x$ has domain $-1 \leqslant x \leqslant 1$ and range $0 \leqslant \cos^{-1} x \leqslant \pi$.

The graph of $y = \cos^{-1} x$, $-1 \leqslant x \leqslant 1$ is obtained by reflecting the graph of $y = \cos x$, $0 \leqslant x \leqslant \pi$, in the line $y = x$.

Warning: $\cos^{-1} x \neq \frac{1}{\cos x}$.



$\cos^{-1}(\cos x) = x$ for $0 \leqslant x \leqslant \pi$,
$\cos(\cos^{-1} x) = x$ for $-1 \leqslant x \leqslant 1$.

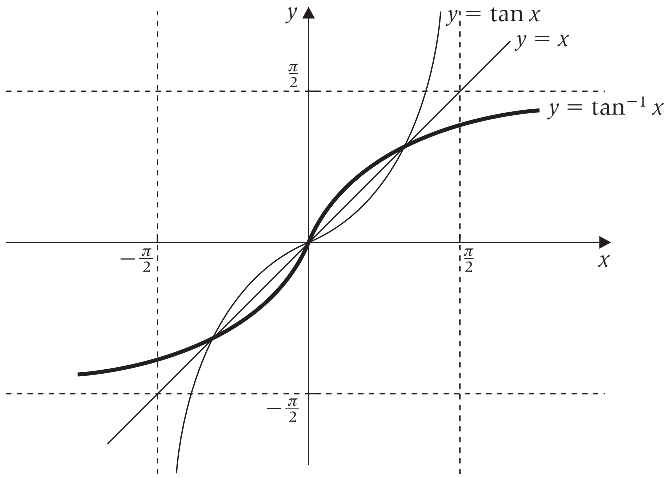
The tangent function can be made one-one by restricting its domain to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The inverse function of $\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, is written as $\tan^{-1} x$.

$y = \tan^{-1} x \Leftrightarrow \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$\tan^{-1} x$ has domain all real numbers and range $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$.

The graph of $y = \tan^{-1} x$, is obtained by reflecting the graph of $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, in the line $y = x$.



Warning: $\tan^{-1} x \neq \frac{1}{\tan x}$.

The lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ are vertical asymptotes of the graph of $y = \tan x$. When reflected in the line $y = x$ they become horizontal.

The lines $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ are horizontal asymptotes of the graph of $y = \tan^{-1}x$.

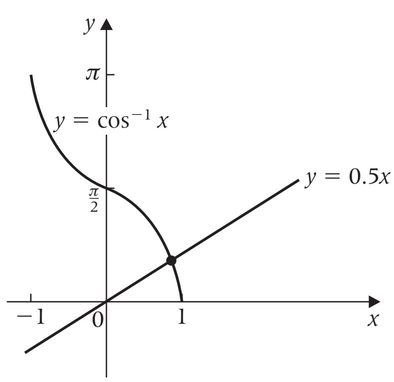
Worked example 3.3

By considering the graphs of $y = 0.5x$ and $y = \cos^{-1} x$ determine the number of real roots of the equation $2 \cos^{-1} x = x$.

Solution

The equation $2 \cos^{-1} x = x$ can be rearranged into the form $\cos^{-1} x = 0.5x$.
The number of real roots of the equation correspond to the number of times the graphs $y = \cos^{-1} x$ and $y = 0.5x$ intersect.

The graphs intersect in just one point so the equation $2 \cos^{-1} x = x$ has only one real root.



EXERCISE 3A

In this exercise you may assume the results in the table opposite. You do not need to learn them for the exam.

Angle θ in radians	**sin θ**	**cos θ**	**tan θ**
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞
π	0	-1	0

1 Find the exact values of:

(a) $\cos^{-1}\frac{1}{2}$ **(b)** $\cos(\cos^{-1}0)$

(c) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ **(d)** $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

(e) $\cos^{-1}\left(\sin\frac{2\pi}{3}\right)$ **(f)** $\cos^{-1}\left(\sin\frac{4\pi}{3}\right)$

2 Find the exact values of:

(a) $\tan^{-1}\frac{1}{\sqrt{3}}$ **(b)** $\tan[\tan^{-1}(-1)]$

(c) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ **(d)** $\tan\left(\cos^{-1}\frac{1}{2}\right)$

(e) $\tan\left(\cos^{-1}\frac{1}{3}\right)$ **(f)** $\sin\left(\tan^{-1}\frac{1}{2}\right)$

3 You are given that $h(x) = \sin^{-1}x + \cos^{-1}x - 3\tan^{-1}x$. Find the value of:

(a) $h(1)$, **(b)** $h(-1)$.

4 Determine the number of real roots of the equation $2\sin^{-1}x = x$.

5 Show that the equation $\tan^{-1}x = kx$, where k is a constant, has only one real root when $k < 0$ and state its value.

6 For very large positive values of the constant k, state the number of real roots of the equation $k\tan^{-1}x = x$.

3.2 Secant, cosecant and cotangent

So far you have worked with three trigonometric ratios, sine, cosine and tangent. The remaining three trigonometric ratios are secant, cosecant and cotangent. They are written as sec x, cosec x and cot x, respectively, and are defined as follows:

$$\sec x = \frac{1}{\cos x}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

Secant is the reciprocal of cosine. Do **not** write sec x as $\cos^{-1}x$, which means something entirely different as you saw in section 3.1. Similarly cosec $x \neq \sin^{-1}x$ and cot $x \neq \tan^{-1}x$.

Used $\tan x = \frac{\sin x}{\cos x}$, from C2 section 6.5.

Calculators do not have the function keys for secant, cosecant and cotangent. The next worked example shows you how to find the values of these functions.

Worked example 3.4

(a) Find the values of:

(i) sec 40°,

(ii) $\operatorname{cosec}\frac{\pi}{5}$,

(iii) cot 250°.

(b) Given that $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, find the exact value of $\cot\frac{5\pi}{6}$.

Solution

(a) **(i)** $\sec 40° = \frac{1}{\cos 40°} = \frac{1}{0.766\,04\ldots} = 1.305$ (to 3 d.p.),

(ii) $\operatorname{cosec}\frac{\pi}{5} = \frac{1}{\sin\frac{\pi}{5}} = \frac{1}{\sin 0.628\,318\ldots}$

$= \frac{1}{0.587\,785\ldots} = 1.701$ (to 3 d.p.).

Set your calculator in radian mode.

(iii) $\cot 250° = \frac{1}{\tan 250°} = \frac{1}{2.747\,47\ldots} = 0.364$ (to 3 d.p.);

(b) $\cot\frac{5\pi}{6} = \frac{1}{\tan\frac{5\pi}{6}} = \frac{1}{\tan\left(\pi - \frac{\pi}{6}\right)} = \frac{1}{-\tan\frac{\pi}{6}} = \frac{1}{-\frac{1}{\sqrt{3}}} = -\sqrt{3}$

Used $\tan(\pi - \theta) = -\tan\theta$.

The next two worked examples show you how to solve some basic trigonometric equations involving a single secant, cosecant or cotangent function.

Worked example 3.5

Solve these equations for $0° \leqslant x < 360°$. Give your answers to one decimal place.

(a) $\sec x = 1.9$, **(b)** $\operatorname{cosec} x = -1.25$.

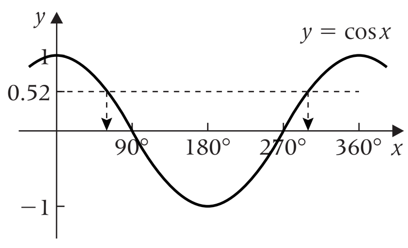
Solution

(a) $\sec x = 1.9 \Rightarrow \cos x = \frac{1}{1.9} = 0.526\,315\ldots$

Cos x is positive so answers lie in the intervals $0° < x < 90°$ and $270° < x < 360°$. The acute angle $\cos^{-1}(0.526\,315\ldots) = 58.24°$,

so $x = 58.24°$ or $x = 360° - 58.24°$

To one decimal place, $x = 58.2°$, $301.8°$.

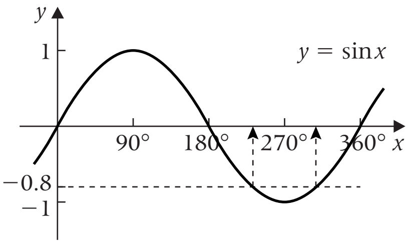


(b) $\text{cosec}\, x = -1.25 \Rightarrow \sin x = -\frac{1}{1.25} = -0.8$

Sin x is negative so answers lie in the interval $180° < x < 360°$. The acute angle, $\sin^{-1} 0.8 = 53.13°$,

so $x = 180° + 53.13°$ or $x = 360° - 53.13°$.

To one decimal place, $x = 233.1°, 306.9°$.



Worked example 3.6

Given that $\tan^{-1} 1 = \frac{\pi}{4}$, solve the equation $\cot 2x = -1$ giving your answer, in terms of π, in the interval for $0 \leqslant x \leqslant 2\pi$.

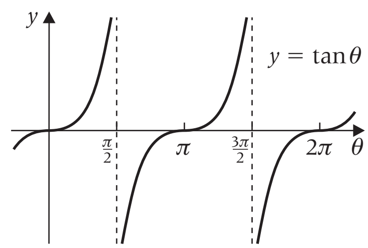
Solution

$\cot 2x = -1 \Rightarrow \tan 2x = -1$.
Since $0 \leqslant x < 2\pi$, you require all values for $2x$ between 0 and 4π.

$\tan^{-1}(-1) = -\frac{\pi}{4}$, and tan is periodic with period π,

$\Rightarrow 2x = \pi - \frac{\pi}{4}$ or $2\pi - \frac{\pi}{4}$ or $3\pi - \frac{\pi}{4}$ or $4\pi - \frac{\pi}{4}$

$\Rightarrow x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}$ and $\frac{15\pi}{8}$.



The next Worked example shows you how to use the definitions of secant, cosecant and cotangent to obtain other trigonometric identities.

Worked example 3.7

Prove the identity $\cot A \sec A \equiv \text{cosec}\, A$.

Solution

$\cot A \sec A \equiv \frac{1}{\tan A} \times \frac{1}{\cos A} \equiv \frac{\cos A}{\sin A} \times \frac{1}{\cos A} \equiv \frac{1}{\sin A} \equiv \text{cosec}\, A$

$\tan A = \frac{\sin A}{\cos A}$

EXERCISE 3B

1 Find the values of:

(a) $\sec 70°$ **(b)** $\text{cosec}\, 70°$

(c) $\cot 20°$ **(d)** $\sec(-70°)$

(e) $\text{cosec}\, 90°$ **(f)** $4 + \cot 430°$

(g) $\frac{1}{1 + \sec 60°}$ **(h)** $\frac{2}{6 + \cot 315°}$

2 Find the values of:

The angles are measured in radians in this question.

(a) sec 2 **(b)** cosec 0.7

(c) cot 0.5 **(d)** sec(−1)

(e) $\operatorname{cosec}\frac{\pi}{8}$ **(f)** $4+\cot\frac{\pi}{8}$

(g) $\dfrac{1}{1+\sec\frac{\pi}{10}}$ **(h)** $\dfrac{1}{6+\cot\frac{\pi}{5}}$

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3 Using the table of results opposite and below, find the exact values of:

(a) sec 60° **(b)** cosec 60°

(c) cot 30° **(d)** sec(−180°)

(e) cosec 135° **(f)** 1 + cot 420°

(g) $\dfrac{1}{\sqrt{3}-\sec 30°}$ **(h)** $\dfrac{2}{7+\sqrt{3}\cot 150°}$

4 Using the table of results opposite, find the exact values of:

(a) $\operatorname{cosec}\frac{\pi}{4}$

(b) $4+\cot\frac{3\pi}{4}$

(c) $\dfrac{1}{1+\sqrt{3}\sec\frac{\pi}{6}}$

(d) $\dfrac{2\sqrt{3}}{2-\cot\frac{\pi}{6}}$

Angle θ in degrees	Angle θ in radians	sin θ	cos θ	tan θ
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
180	π	0	−1	0

5 Solve these equations for $0° \leqslant x \leqslant 360°$.
Give your answers to one decimal place.

(a) $\sec x = 1.8$ **(b)** $\operatorname{cosec} x = -2.25$

(c) $\cot x = 3$ **(d)** $\sec x = -1.3$

(e) $\operatorname{cosec} x = 3$ **(f)** $\cot x = -2.4$

(g) $4\sec 2x = -7$ **(h)** $5\cot 2x = -2$

6 Solve these equations for $0 \leqslant x \leqslant 2\pi$, giving your answers in radians to three significant figures.

(a) $\sec x = 2$ **(b)** $\operatorname{cosec} x = -2$

(c) $\cot 2x = 1$ **(d)** $\sec 5x = -1$

(e) $\sqrt{3}\operatorname{cosec} 3x = 2$ **(f)** $\cot 2x = -\sqrt{3}$

(g) $\sec 3x = 1$ **(h)** $\sqrt{12}\cot 3x = 2$

7 Prove the following identities:

(a) $\tan A \operatorname{cosec} A \equiv \sec A$ **(b)** $\sin A \cot A \equiv \cos A$

(c) $\dfrac{\cot A}{1 + \cot A} \equiv \dfrac{1}{1 + \tan A}$ **(d)** $\tan A \operatorname{cosec} A \equiv \sec A$

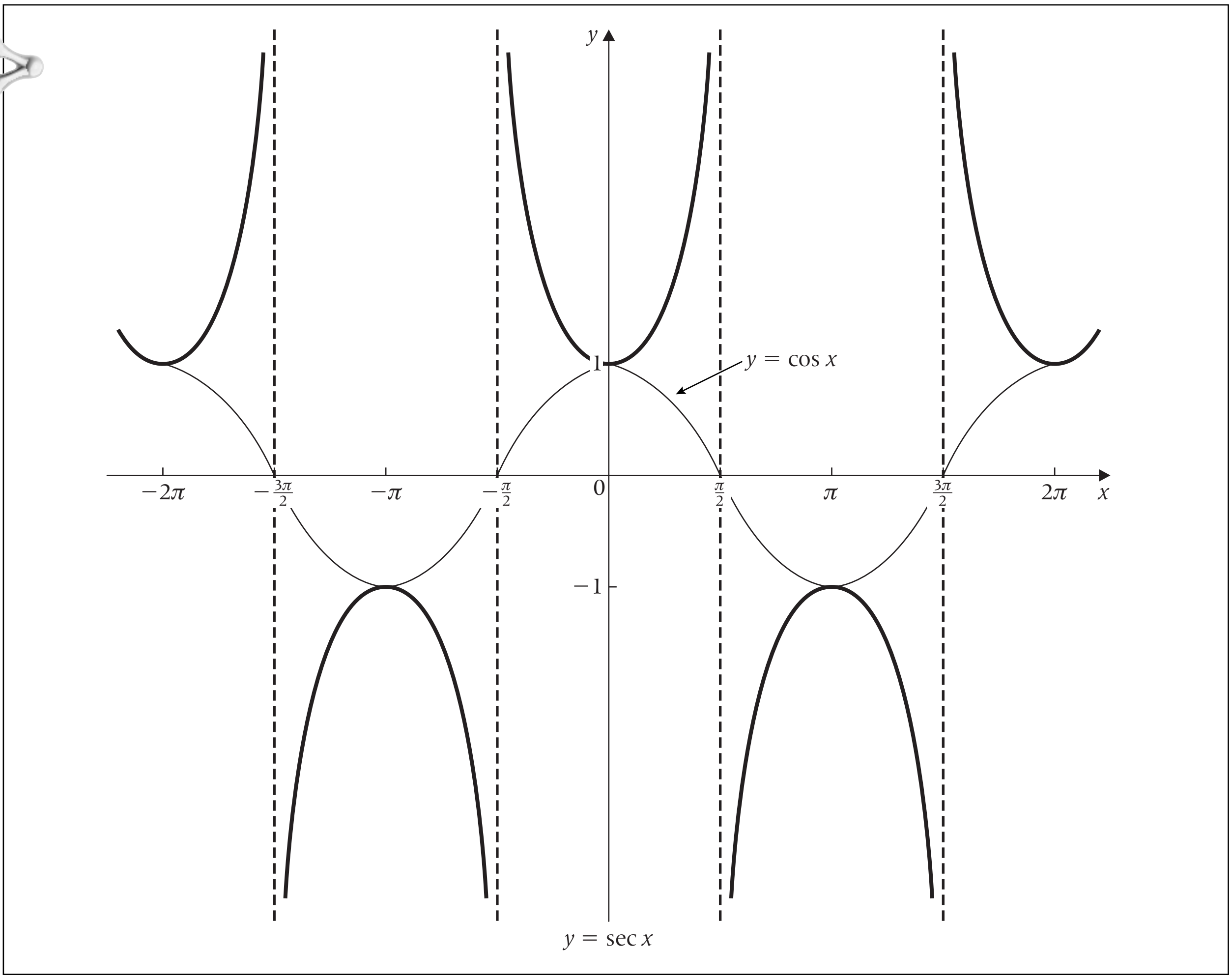
(e) $\sec A - \cos A \equiv \tan A \sin A$

(f) $\dfrac{\sin A}{1 - \cos A} \equiv \operatorname{cosec} A + \cot A$

Hint for part (f): Consider $(1 - \cos A)(\operatorname{cosec} A + \cot A)$.

3.3 Graphs of sec *x*, cosec *x* and cot *x*

Graph of *y* = sec *x*



Comparing the graph of $y = \sec x$ with the graph of $y = \cos x$ you can see that $y = \sec x$ has maximum points where $y = \cos x$ has minimum points and has minimum points where $y = \cos x$ has maximum points.
Secant, like cosine, is a periodic function with period 2π and so the graph repeats itself every 2π radians.

The domain of $\sec x$ is all real values $\neq (2k + 1)\dfrac{\pi}{2}$ and its range is all real values **except** $-1 < y < 1$.

Graph of y = cosec x

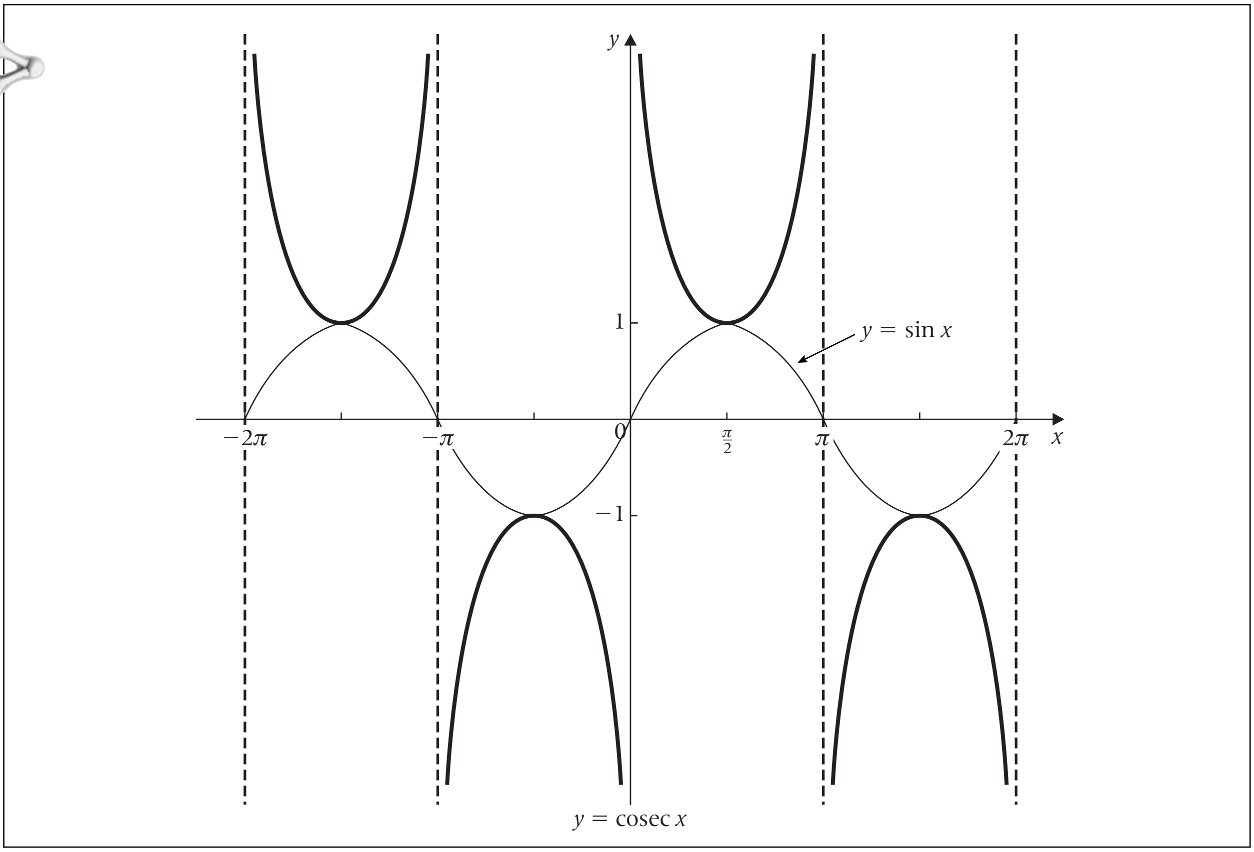
Consider $y = \operatorname{cosec} x = \dfrac{1}{\sin x} = \dfrac{1}{\cos\left(x - \dfrac{\pi}{2}\right)}$

$\Rightarrow \operatorname{cosec} x = \sec\left(x - \dfrac{\pi}{2}\right)$

So, if $f(x) = \sec x$, then $f\left(x - \dfrac{\pi}{2}\right) = \operatorname{cosec} x$.

So a translation of $\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$ transforms the graph of $y = \sec x$ into the graph of $y = \operatorname{cosec} x$.

C2, section 5.2.



Comparing the graph of $y = \operatorname{cosec} x$ with the graph of $y = \sin x$ you can see that $y = \operatorname{cosec} x$ has maximum points where $y = \sin x$ has minimum points and has minimum points where $y = \sin x$ has maximum points.

Cosecant, like sine, is a periodic function with period 2π and so the graph repeats itself every 2π radians.

The domain of $\operatorname{cosec} x$ is all real values $\neq k\pi$ and its range is all real values **except** $-1 < y < 1$.

Graph of y = cot x

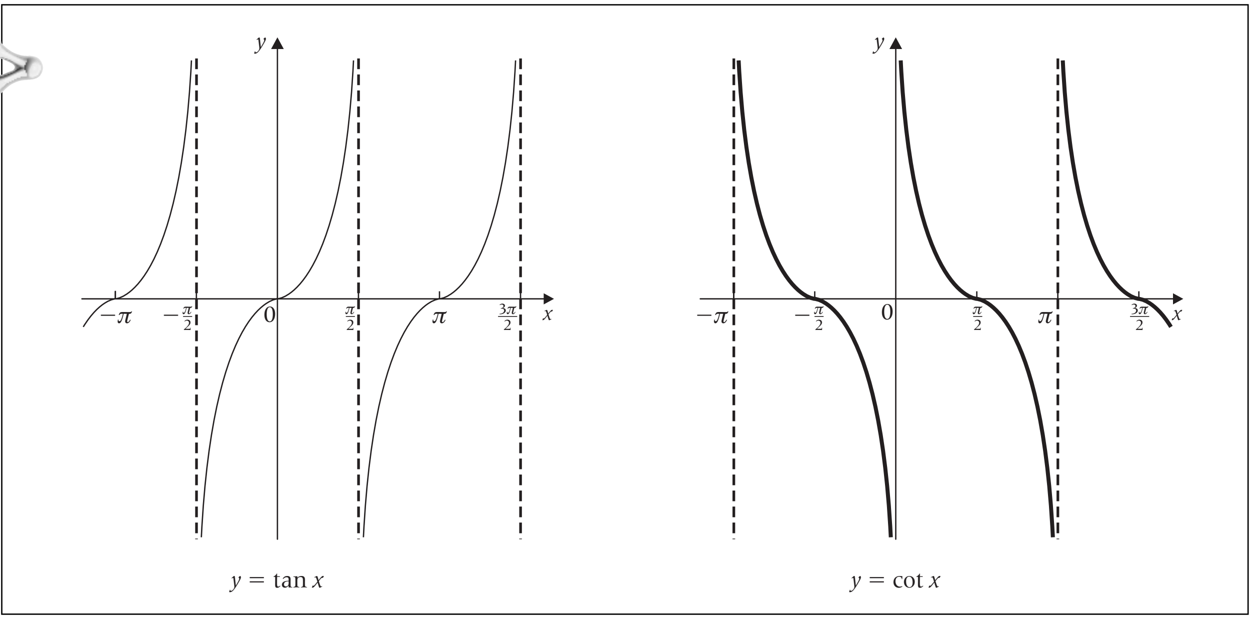
Consider $y = \cot x = \dfrac{\cos x}{\sin x} = \dfrac{\sin\left(\dfrac{\pi}{2} - x\right)}{\cos\left(\dfrac{\pi}{2} - x\right)}$

$\Rightarrow \cot x = \tan\left(\dfrac{\pi}{2} - x\right) = \tan\left[2\left(\dfrac{\pi}{4}\right) - x\right]$

So, if f(x) = tan x, then $f\left[2\left(\dfrac{\pi}{4}\right) - x\right] = \cot x$.

So a reflection in the line $x = \dfrac{\pi}{4}$ transforms the graph of $y = \tan x$ into the graph of $y = \cot x$.

C3 Ex2B Q16(c).



Cotangent, like tangent, is a periodic function with period π and so the graph repeats itself every π radians.
The domain of cot x is all real values $\neq k\pi$ and its range is all real values.

Worked example 3.8

(a) Determine the transformation that maps $y = \sec x$ onto $y = \sec 2x$.

(b) State the period, in radians, of the graph $y = \sec 2x$.

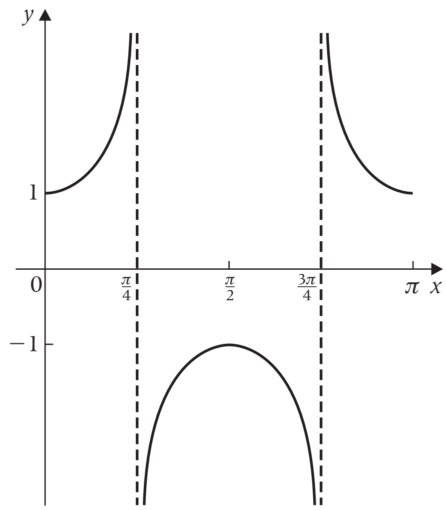
(c) Sketch the graph of $y = \sec 2x$ for $0 \leqslant x \leqslant \pi$, $x \neq \dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

Solution

(a) Let f(x) = sec x then f($2x$) = sec $2x$.
The transformation that maps $y = \sec x$ onto $y = \sec 2x$ is a stretch of scale factor $\dfrac{1}{2}$ in the x-direction.

(b) The period of $y = \sec 2x$ is $\frac{2\pi}{2} = \pi$ radians.

(c) The graph of $y = \sec 2x$ for $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{4}, \frac{3\pi}{4}$ is



EXERCISE 3C

1 **(a)** Determine the transformation that maps $y = \operatorname{cosec} x$ onto $y = \operatorname{cosec} 4x$.

(b) State the period, in radians, of the graph $y = \operatorname{cosec} 4x$.

2 **(a)** Determine the transformation that maps $y = \cot x$ onto $y = \cot \frac{x}{4}$.

(b) State the period, in radians, of the graph $y = \cot \frac{x}{4}$.

3 Sketch the graph of $y = 1 + \sec \frac{x}{2}$ for $-\pi < x < \pi$.

4 **(a)** Sketch the graph of $y = 1 + 3 \operatorname{cosec} 2x$ for $-180° < x < 180°$, $x \neq 0°, \pm 90°$.

(b) Describe a sequence of transformations that maps the curve $y = \sec x$ onto the curve $y = 1 + 3 \operatorname{cosec} 2x$.

3.4 Identities involving the squares of secant, cosecant and cotangent

In C2, section 6.4, you used the identity $\cos^2\theta + \sin^2\theta \equiv 1$. In this section you will be shown two similar identities.

If you divide each term in the identity $\cos^2\theta + \sin^2\theta \equiv 1$ by $\cos^2\theta$ you get

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

$\Rightarrow 1 + \tan^2 \theta \equiv \sec^2 \theta$

Similarly, if you divide each term in the identity $\cos^2 \theta + \sin^2 \theta \equiv 1$ by $\sin^2 \theta$, you get

$$\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} \equiv \frac{1}{\sin^2 \theta}$$

$\Rightarrow \cot^2 \theta + 1 \equiv \operatorname{cosec}^2 \theta$

These two identities, along with $\cos^2 \theta + \sin^2 \theta \equiv 1$, are frequently used to solve trigonometric equations and to prove other identities. They are **not** given in the examination formulae booklet so you must memorise them and know how to apply them.

Worked example 3.9

Solve the equation $\sec^2 x = 4 + 2 \tan x$, giving all solutions in the interval $0° \leqslant x < 360°$.

Solution

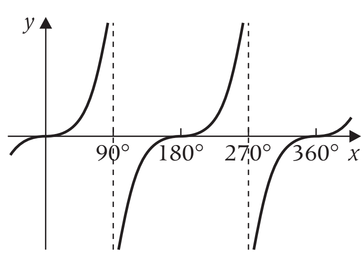
To solve the equation $\sec^2 x = 4 + 2 \tan x$
use the identity $\sec^2 x \equiv 1 + \tan^2 x$ to get $1 + \tan^2 x = 4 + 2 \tan x$.

As a starter, try to get a quadratic equation in the non-squared trigonometric term, tan *x* here.

$\Rightarrow \tan^2 x - 2 \tan x - 3 = 0$

$\Rightarrow (\tan x - 3)(\tan x + 1) = 0$

$\Rightarrow \tan x = 3, \tan x = -1.$



For $\tan x = 3$, $\tan^{-1} 3 = 71.57°$.

For $0° \leqslant x < 360°$, $\tan x = 3 \Rightarrow x = 71.57°, 180° + 71.57°$

For $\tan x = -1$, $\tan^{-1}(-1) = -45°$.

For $0° \leqslant x < 360°$, $\tan x = -1 \Rightarrow x = 180° - 45°, 360° - 45°$

The solutions of the equation $\sec^2 x = 4 + 2 \tan x$, in the interval $0° \leqslant x < 360°$, are $x = 71.6°, 135°, 251.6°, 315°$.

The general strategy when asked to solve trigonometric equations which involve the same angle but with a squared trigonometric function is to try to write the equation as a quadratic equation in the non-squared trigonometric function.

Worked example 3.10

Solve the equation $\operatorname{cosec}^2 x = 5 + 3\cot x$, giving all solutions in the interval $0 \leqslant x < 2\pi$, to three significant figures.

Solution

To solve the equation $\operatorname{cosec}^2 x = 5 + 3\cot x$
use the identity $\operatorname{cosec}^2 x \equiv 1 + \cot^2 x$ to get $1 + \cot^2 x = 5 + 3\cot x$

Try to write the equation as a quadratic equation in cot *x*.

$\Rightarrow \cot^2 x - 3\cot x - 4 = 0$

$\Rightarrow (\cot x - 4)(\cot x + 1) = 0$

$\Rightarrow \cot x = 4, \cot x = -1$

$\Rightarrow \tan x = \dfrac{1}{4}, \tan x = -1$

Used $\tan x = \dfrac{1}{\cot x}$.

For $\tan x = \dfrac{1}{4}$, $\tan^{-1}\dfrac{1}{4} = 0.245$ rads.

Set calculator in radian mode.

For $0 \leqslant x < 2\pi$, $\tan x = \dfrac{1}{4} \Rightarrow x = 0.245$ rads, $(\pi + 0.245)$ rads.

For $\tan x = -1$, $\tan^{-1}(-1) = -0.785$ rads.
For $0 \leqslant x < 2\pi$,
$\tan x = -1 \Rightarrow x = (\pi - 0.785)$ rads, $(2\pi - 0.785)$ rads.

The solutions of the equation $\operatorname{cosec}^2 x = 5 + 3\cot x$, in the interval $0 \leqslant x < 2\pi$, are 0.245^c, 2.36^c, 3.39^c and 5.50^c, to 3 sf.

Worked example 3.11

Prove the identity $\sec^2 A - \operatorname{cosec}^2 A \equiv (\tan A + \cot A)(\tan A - \cot A)$.

Solution

$$\sec^2 A - \operatorname{cosec}^2 A \equiv 1 + \tan^2 A - (1 + \cot^2 A)$$
$$\equiv \tan^2 A - \cot^2 A$$
$$\equiv (\tan A + \cot A)(\tan A - \cot A)$$

Difference of two squares.

So $\sec^2 A - \operatorname{cosec}^2 A \equiv (\tan A + \cot A)(\tan A - \cot A)$.

EXERCISE 3D

1 Prove the identity
$\tan^2 A - \cot^2 A \equiv (\sec A + \operatorname{cosec} A)(\sec A - \operatorname{cosec} A)$.

2 Prove the identity
$\cot^2 A + \sin^2 A \equiv (\operatorname{cosec} A + \cos A)(\operatorname{cosec} A - \cos A)$.

3 Prove the identity $\operatorname{cosec}^2 A \cos^2 A \equiv \operatorname{cosec}^2 A - 1$.

4 Given that $2\cos^2 x - \sin^2 x = 1$, show that $\cos^2 x = 2\sin^2 x$ and hence find the possible values of $\cot x$.

5 Given that $5\sec^2 x + 3\tan^2 x = 9$, find the possible values of $\sin x$.

6 Given that $x = \sec\theta$ and $y = \tan\theta$, show that $x^2 - y^2 = 1$.

7 Prove the identity

$$\frac{(\sec A - \tan A)(\tan A + \sec A)}{\operatorname{cosec} A - \cot A} \equiv \cot A + \operatorname{cosec} A.$$

8 Prove the identity $(\operatorname{cosec} A + \cot A)^2 \equiv \dfrac{1 + \cos A}{1 - \cos A}$.

9 Eliminate θ from equations $x = \operatorname{cosec}\theta$ and $y = \dfrac{1}{4}\cot\theta$.

10 Eliminate θ from equations $x = 2 + \operatorname{cosec}\theta$ and $y = \dfrac{1}{4}\tan\theta$.

11 Solve the equation $\sec^2 x = 3 + \tan x$, giving all solutions in the interval $0° \leqslant x < 360°$.

12 Solve the equation $3\sec^2 x = 5 + \tan x$, giving all solutions in the interval $-180° \leqslant x < 180°$.

13 Solve the equation $2\operatorname{cosec}^2 x = 1 + 3\cot x$, giving all solutions in the interval $0° \leqslant x < 360°$.

14 Solve the equation $\tan^2 x = 1 - \sec x$, giving all solutions in the interval $-180° \leqslant x < 180°$.

15 Solve the equation $\cot^2 x + \operatorname{cosec} x = 11$, giving all solutions in the interval $0° \leqslant x < 360°$.

16 Solve the equation $4\tan^2 x + 12\sec x + 1 = 0$, giving all solutions in the interval $0° \leqslant x < 360°$.

17 Solve the equation $\cot^2 x + \operatorname{cosec}^2 x = 7$, giving all solutions in the interval $0 \leqslant x < 2\pi$.

18 Solve the equation $\tan^2 x + 3 = 3\sec x$, giving all solutions in the interval $0 \leqslant x < 2\pi$.

19 Solve the equation $\cot^2 x + 5\operatorname{cosec} x = 3$, giving all solutions in the interval $0 \leqslant x < 2\pi$.

20 Solve the equation $2\operatorname{cosec}^2 2x + \cot 2x = 3$, giving all solutions in the interval $-\pi \leqslant x < \pi$.

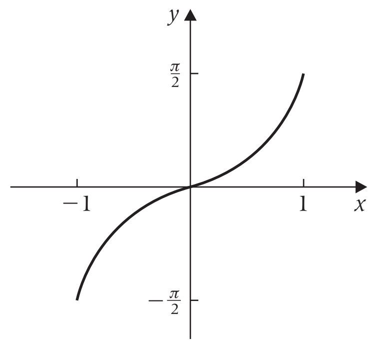
21 Given that $x = \sec A + \tan A$, show that $x + \dfrac{1}{x} = 2\sec A$.

22 Given that $x = \operatorname{cosec}\theta + \cot\theta$, show that $x + \dfrac{1}{x} = 2\operatorname{cosec}\theta$.

Key point summary

1 $\sin^{-1} x$ has domain $-1 \leqslant x \leqslant 1$ and range $-\frac{\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$. *p43*

The graph of $y = \sin^{-1} x$, $-1 \leqslant x \leqslant 1$ is obtained by reflecting the graph of $y = \sin x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, in the line $y = x$.



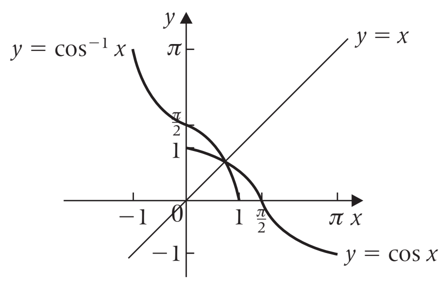
Warning: $\sin^{-1} x \neq \dfrac{1}{\sin x}$.

2 $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, *p43*

$\sin(\sin^{-1} x) = x$ for $-1 \leqslant x \leqslant 1$.

3 $\cos^{-1} x$ has domain $-1 \leqslant x \leqslant 1$ and range $0 \leqslant \cos^{-1} x \leqslant \pi$. *p44*

The graph of $y = \cos^{-1} x$, $-1 \leqslant x \leqslant 1$ is obtained by reflecting the graph of $y = \cos x$, $0 \leqslant x \leqslant \pi$, in the line $y = x$.



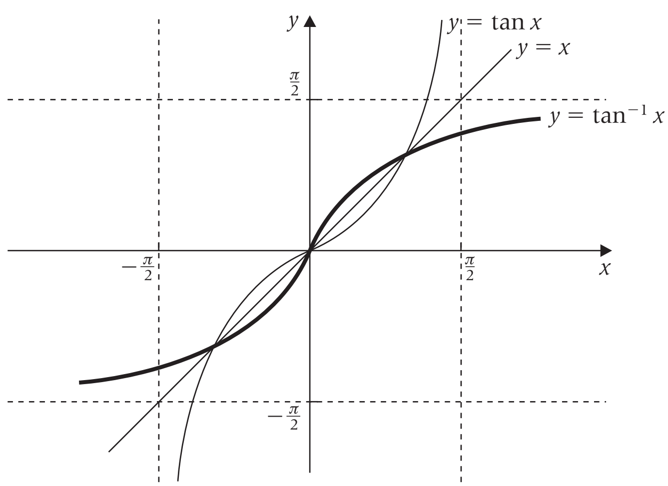
Warning: $\cos^{-1} x \neq \dfrac{1}{\cos x}$.

4 $\cos^{-1}(\cos x) = x$ for $0 \leqslant x \leqslant \pi$, *p44*

$\cos(\cos^{-1} x) = x$ for $-1 \leqslant x \leqslant 1$.

5 $\tan^{-1} x$ has domain all real numbers and range $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$. *p45*

The graph of $y = \tan^{-1} x$, is obtained by reflecting the graph of $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, in the line $y = x$.



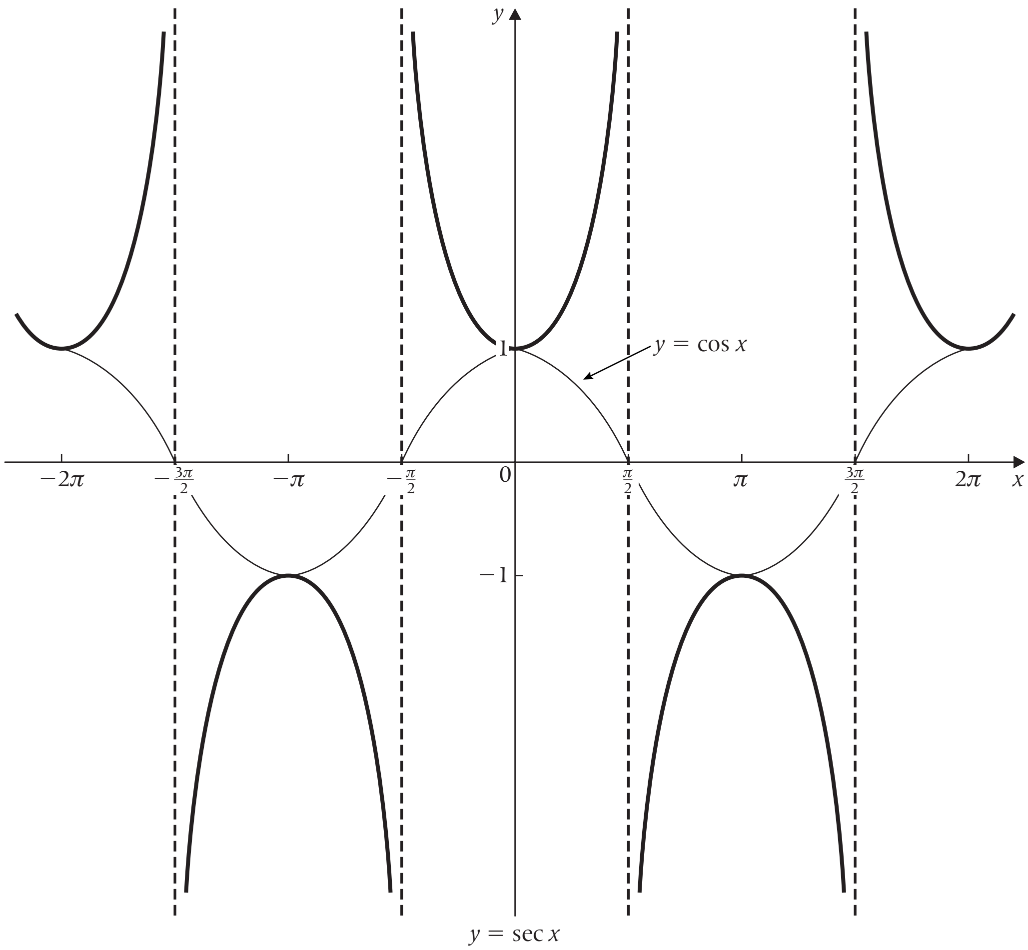
Warning: $\tan^{-1} x \neq \dfrac{1}{\tan x}$.

6 $\sec x = \dfrac{1}{\cos x}$ *p46*

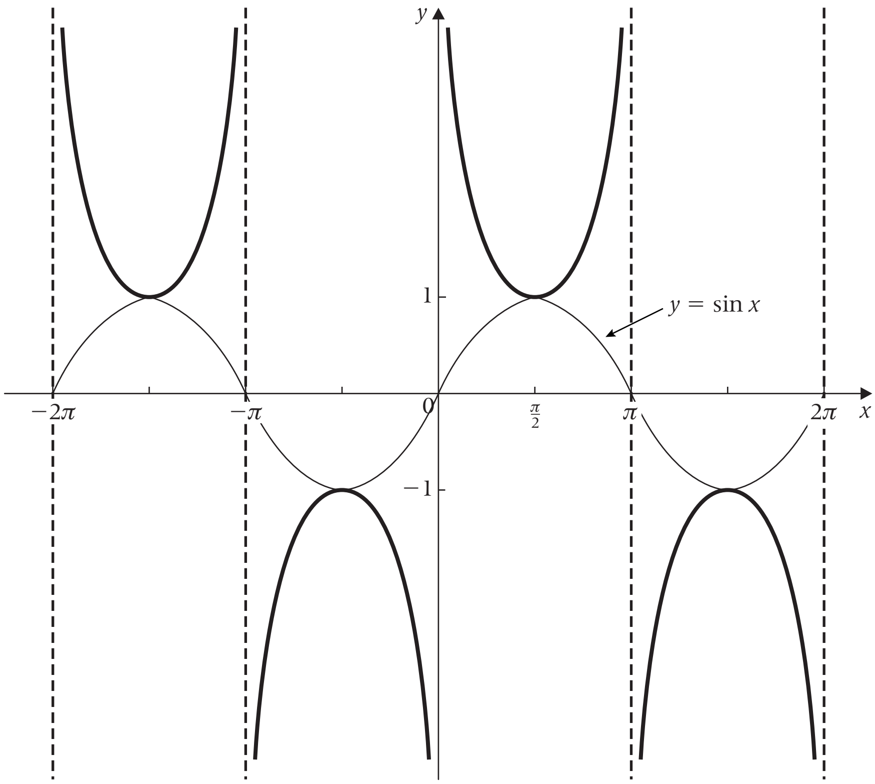
$\operatorname{cosec} x = \dfrac{1}{\sin x}$

$\cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

7 *p50*

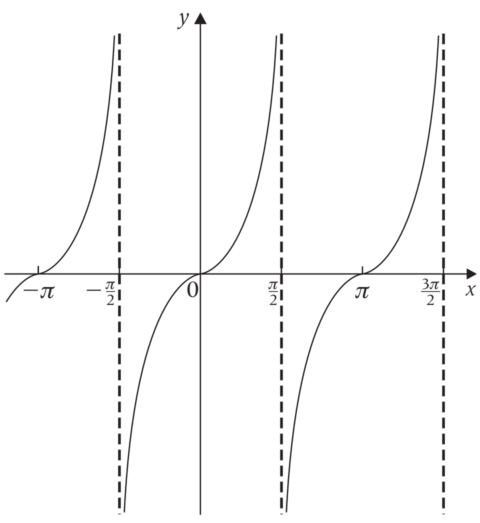


8 *p51*

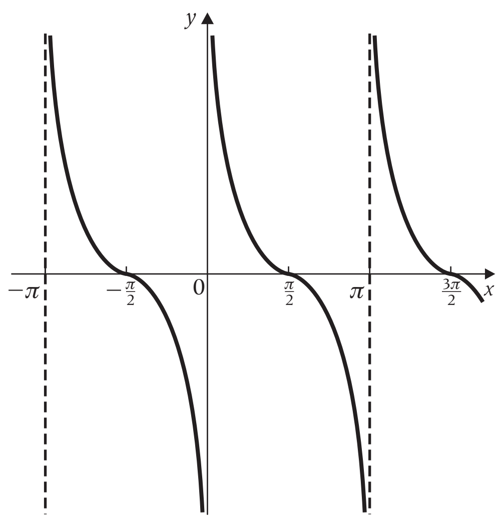


$y = \operatorname{cosec} x$

9 *p52*



$y = \tan x$



$y = \cot x$

10 $1 + \tan^2\theta \equiv \sec^2\theta$ *p54*

11 $\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$ *p54*

Test yourself	What to review
1 Show that the equation $\cos^{-1} x = \sin^{-1} x$ has only one real root and state its value to three significant figures.	*Section 3.1*
2 Solve the equation $\text{cosec}(x + 30°) = 2.1$, for $0° \leqslant x < 360°$. Give your answers to one decimal place.	*Section 3.2*
3 By sketching the graphs of $y = \cot x$ and $y = \dfrac{x}{3}$ for $0 < x \leqslant \dfrac{\pi}{2}$, show that the equation $\cot x = \dfrac{x}{3}$ has one root and explain why this root must be in the interval $1 < x \leqslant \dfrac{\pi}{2}$.	*Section 3.3*
4 Solve the equation $3\tan^2 x + 2\sec x = 5$, giving all solutions in the interval $0 \leqslant x < 2\pi$.	*Section 3.4*

Test yourself ANSWERS

1 0.707.

2 121.6°, 358.4°.

4 $x = 0.723^c, 2.09^c, 4.19^c, 5.56^c$.